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A MICROPOLAR THEORY OF FINITE DEFORMATION AND FINITE ROTATION MULTIPLICATIVE ELASTOPLASTICITY

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Abstract—The aim of this work is to formulate a geometrically exact theory of finite deformation and finite rotation micropolar elastoplasticity to obtain a generalized nonlinear continuum framework. To this end, the *classical* deformation map is supplemented by an *independent* rotation field to yield an enhanced configuration space. Thereby, the rotational part of the formulation is consequently parameterized in terms of the rotation (pseudo) vector via the Euler–Rodrigues formula. Then, micropolar hyperelasticity and *multiplicative* elastoplasticity are conceptionally derived as in the *classical* Boltzmann continuum. The proposed theory is consequently developed in a modern geometry oriented fashion. Linearization of the kinematics retrofits the well-known structure of the micropolar geometrically linear theory.

1. INTRODUCTION

Generalized continuum descriptions involving *independent* rotations have been introduced by the brothers Cosserat (1909) at the beginning of this century. Further emphasis was directed to this and related theories by a community of researchers several decades later among whom we find such prominent members as Günther (1958), Koiter (1964), Mindlin (1964), Toupin (1964), Neuber (1966), Schaefer (1967), Eringen (1968), Lippmann (1969), Besdo (1974), Reissner (1987) and references therein. These researchers were mainly attracted by the theoretical challenges and beauties of nonconventional continuum theories. Large-scale numerical computations have not been in the scope of those approaches. In comparison to the application within continuum mechanics the Cosserat approach has been adopted more extensively to nonlinear models in structural mechanics. Geometrically exact models for rods, plates and shells have often been referred to as one-director Cosserat lines and surfaces. Within those formulations the Cosserat ideas have been compiled to modern mechanics incorporating finite rotations and finite strains, notably the works on rods and shells by Simo and Vu-Quoc (1986) and Simo and Fox (1989) and references therein.

Recently, renewed interest in micropolar continua arose within the context of localization computations. Today it is fairly well understood that numerical modelling of materials exhibiting strain softening or nonsymmetric material operators due to non-associated flow, for example, leads to a pathological mesh dependence of the post-peak response within the *classical* continuum description when the deformation pattern obeys a highly localized zone. Researchers looking for a remedy for this deficiency revived the micropolar approach since it turned out that rotations are an essential ingredient in failure bands where shear failure mechanisms play a dominant role. This new approach to regularize the mesh sensitivity of localization computations was perused mainly by Mühlhaus and Vardoulakis (1987), Mühlhaus (1989), de Borst (1991, 1993), de Borst and Mühlhaus (1992), Steinmann and William (1991) and Dietsche *et al.* (1993).

Nevertheless, these authors argued either along the lines of a geometrical linearized description or adopted two-dimensional rate formulations in the sense of hypoelasticity together with a straightforward translation of the small strain and curvature flow rules to the nonlinear regime. Motivated by the results of the works cited above and in extension of known formulations it is the aim of this paper to develop a fully nonlinear continuum theory of geometrically exact finite deformation and finite rotation micropolar elastoplasticity. To this end, the terminology in modern geometry oriented continuum mechanics is used whenever possible to ease comparison with the *classical* continuum description.

For a comprehensive account on the terminology of *push-forward* and *pull-back* or the Lie derivative of spatial tensor fields within *classical* nonlinear continuum mechanics see the textbook by Marsden and Hughes (1983), for example.

The novel results of the approach advocated in this contribution are: the theoretical framework of a fully three-dimensional nonlinear continuum theory invoking *independent* rotations, the consideration of large strains together with large rotations parameterized by the rotation (pseudo) vector, the reduction of the originally third-order curvature tensors to second-order curvature measures, the postulate of a stored energy function to achieve micropolar hyperelasticity, the introduction of a geometrically exact *multiplicative* format of elastoplasticity, the derivation of associated nonstandard flow rules and finally the embedding of the theory within a variational principle.

An outline of the paper is as follows: first, the enhanced configuration space is defined and different strain and curvature measures are introduced. Then stress and couple stress measures are derived by resorting to the Cauchy theorem and representations with respect to different configurations are established. After postulating *multiplicative* decompositions of the deformation gradient and the *independent* rotation tensor into an elastic and an inelastic part, different configurations associated with these decompositions are defined, and strain and curvature measures are connected by *push-forward* and *pull-back* operations. To finalize the kinematical description, rates of the strain and curvature measures are examined in detail. For the sake of comparison with the conventional Boltzmann continuum, rates of the stress and couple stress measures are developed in the sequel. Next, the hyperelastic part of the stress and couple stress response is formulated by postulating an isotropic stored energy function in terms of the nonsymmetric strain and curvature tensors. Then the associated flow rules for the plastic parts of the deformation gradient and the micropolar rotation are derived by exploiting the Clausius–Duhem inequality and the postulate of maximum dissipation. Finally, issues concerning the formulation of the balance equations and the micropolar Dirichlet variational principle together with its Hesse matrix are addressed.

2. MICROPOLAR NONLINEAR KINEMATICS

2.1. *Configuration space*

Let $\mathscr{B}_0 \subset \mathbb{R}^3$ be the reference configuration of a material body with boundary $\partial\mathscr{B}_0$ and introduce the differentiable nonlinear deformation map $\boldsymbol{\varphi}(\mathbf{X}): \mathscr{B}_0 \to \mathbb{R}^3$ taking particles labelled by their position $\mathbf{X}$ in the reference configuration to their placement $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$ in the spatial configuration $\mathscr{B}$. It is assumed throughout that the deformation map possesses an invertible linear tangent map $\mathbf{F} = \nabla_X \boldsymbol{\varphi}$, denoted by the deformation gradient, with $J = \det \mathbf{F} > 0$. Within the concept of convected coordinates the deformation gradient maps base vectors $\mathbf{G}_I$ defined in the tangent space $T\mathscr{B}_0$ to base vectors $\mathbf{g}_i$ in the tangent space $T\mathscr{B}$ and therefore allows the representation $\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^I$.

Throughout this paper capital and lower-case bold-face letters denote vectors or tensors referred to the reference or the spatial configuration, respectively. Equivalently, capital or lower-case indices are attached to material or spatial quantities with the understanding that $i \equiv I$.

The enhanced continuum representation is obtained by introducing an additional *independent* rotation field $\bar{\mathbf{R}}: \mathscr{B}_0 \to SO(3)$ defining the orientation of a separate triade of base vectors $\bar{\mathbf{g}}_i$ attached to each material point $\mathbf{X}$ which rotates *independently* with respect to the material triade of base vectors $\mathbf{G}^I$. Equivalently, we may say that the *independent* rotation field $\bar{\mathbf{R}}: \mathscr{B} \to SO(3)$ defines the orientation of an additional triade of base vectors $\bar{\mathbf{G}}^I$ attached to the image $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X})$ of each material point which rotates *independently* with respect to the spatial triade of base vectors $\mathbf{g}_i$. Therefore, we have the representation for the *independent* micropolar rotation

$$\bar{\mathbf{R}} = \bar{\mathbf{g}}_i \otimes \mathbf{G}^I = \mathbf{g}_i \otimes \bar{\mathbf{G}}^I. \tag{1}$$

Then the enhanced configuration space of a micropolar continuum is introduced as

$$\mathscr{C} = \{(\boldsymbol{\varphi}, \bar{\mathbf{R}}) : \mathscr{B}_0 \to \mathbb{R}^3 \times \mathrm{SO}(3)\}. \tag{2}$$

Here SO(3) denotes the group of proper orthogonal transformations.

Remark. The field of orthogonal transformations $\bar{\mathbf{R}} = \exp(\mathrm{spn}(\boldsymbol{\theta}))$ is parameterized by the *independent* rotation (pseudo) vector field $\boldsymbol{\theta}: \mathscr{B}_0 \to \mathbb{R}^3$. Then $\boldsymbol{\theta}$ defines the axis of rotation with the rotation angle $\theta = \|\boldsymbol{\theta}\|$ and constitutes the only eigenvector with (real) unit eigenvalue for $\bar{\mathbf{R}} \cdot \boldsymbol{\theta} = \boldsymbol{\theta}$ and the only eigenvector with (real) zero eigenvalue for $\mathrm{spn}(\boldsymbol{\theta}) \cdot \boldsymbol{\theta} = \mathbf{0}$. The exponential map

$$\exp(\cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} (\cdot)^n : \mathrm{so}(3) \to \mathrm{SO}(3)$$

is expressed in closed form by the Euler–Rodrigues formula

$$\exp(\mathrm{spn}(\boldsymbol{\theta})) = \cos\theta\, \mathbf{I} + \frac{\sin\theta}{\theta} \mathrm{spn}(\boldsymbol{\theta}) + \frac{1-\cos\theta}{\theta^2} \boldsymbol{\theta} \otimes \boldsymbol{\theta}. \tag{3}$$

Here so(3) denotes the set of skew-symmetric tensors. With the definition of the isotropic third-order Ricci-tensor (permutation tensor) together with its relation to the symmetric second order unit tensor $\mathbf{I}$ and the skew-symmetric fourth-order unit tensor $\mathscr{I}^{\mathrm{skw}}$

$$\overset{3}{\mathbf{e}} = [\mathbf{g}_i \mathbf{g}_j \mathbf{g}_k] \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \quad \text{with} \quad \overset{3}{\mathbf{e}} : \overset{3}{\mathbf{e}} = 2\mathbf{I} \quad \text{and} \quad \overset{3}{\mathbf{e}} \cdot \overset{3}{\mathbf{e}} = 2\mathscr{I}^{\mathrm{skw}} \tag{4}$$

the skew-symmetric tensor associated with an axial vector follows as

$$\mathrm{spn}(\boldsymbol{\theta}) = -\overset{3}{\mathbf{e}} \cdot \boldsymbol{\theta} \in \mathrm{so}(3) \tag{5}$$

and vice versa the axial vector associated with a skew-symmetric tensor is extracted as

$$\mathrm{axl}(\mathrm{spn}(\boldsymbol{\theta})) = -\tfrac{1}{2}\overset{3}{\mathbf{e}} : \mathrm{spn}(\boldsymbol{\theta}) = \tfrac{1}{2}\overset{3}{\mathbf{e}} : \overset{3}{\mathbf{e}} \cdot \boldsymbol{\theta} = \boldsymbol{\theta} \in \mathbb{R}^3. \tag{6}$$

Example. Define the standard basis $\mathbf{E}_I$ in $\mathbb{R}^3$, and consider an in-plane rotation about the $\mathbf{E}_3$-axis with $\boldsymbol{\theta} = \theta \mathbf{E}_3$. Then $\mathrm{spn}(\boldsymbol{\theta})$ and $\exp(\mathrm{spn}(\boldsymbol{\theta}))$ have the matrix representation relative to $\mathbf{E}_I$

$$\mathrm{spn}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \exp(\mathrm{spn}(\boldsymbol{\theta})) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

2.2. *Micropolar strain*

The motivation for the choice of the micropolar strain measures is the definition of the *classical* right and left stretch tensors $\mathbf{U} = \mathbf{R}^t \cdot \mathbf{F}$ and $\mathbf{v} = \mathbf{F} \cdot \mathbf{R}^t$, where $\mathbf{R}$ denotes the continuum rotation. Consequently, *multiplicative* decompositions of the deformation gradient $\mathbf{F}$ into the *independent* micropolar rotation $\bar{\mathbf{R}}$ and the nonsymmetric micropolar right $\bar{\mathbf{U}}$ respectively left $\bar{\mathbf{v}}$ stretch tensor are introduced within the setting of micropolar continuum theory as

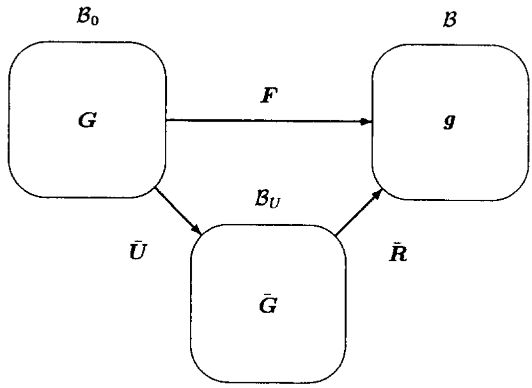

Fig. 1. Micropolar decomposition of the deformation gradient.

$$\mathbf{F} = \bar{\mathbf{R}} \cdot \bar{\mathbf{U}} = \bar{\mathbf{v}} \cdot \bar{\mathbf{R}} \tag{8}$$

(see Fig. 1). Thereby, the micropolar stretch tensors $\bar{\mathbf{U}}$ and $\bar{\mathbf{v}}$ contain the difference between the continuum and the micropolar rotation $\bar{\mathbf{R}}^t \cdot \mathbf{R}$ in addition to the *classical* stretch tensors $\mathbf{U}$ and $\mathbf{v}$. In analogy to the Cauchy–Green and Finger tensors $\mathbf{C}$ and $\mathbf{b}$ of the Boltzmann continuum, the micropolar stretch tensors may be conceived as the *pull-back* (respectively the *push-forward*) of the spatial material metric tensors $\mathbf{g}$ and $\mathbf{G}^{-1}$

$$\bar{\mathbf{U}} = \bar{\mathbf{R}}^t \cdot \mathbf{g} \cdot \mathbf{F} \quad \text{and} \quad \bar{\mathbf{v}} = \mathbf{F} \cdot \mathbf{G}^{-1} \cdot \bar{\mathbf{R}}^t. \tag{9}$$

Expressing the strain measures with respect to the convective base vectors highlights their relation to the metric tensors by *pull-back* and *push-forward* operations through $\mathbf{F}$ and $\bar{\mathbf{R}}$

$$\bar{\mathbf{U}} = g_{ij} \bar{\mathbf{G}}^I \otimes \mathbf{G}^J \quad \text{and} \quad \bar{\mathbf{v}} = G^{IJ} \mathbf{g}_i \otimes \bar{\mathbf{g}}_j. \tag{10}$$

Under a superposed rigid body motion with $\mathbf{x}^* = \mathbf{c} + \mathbf{Q} \cdot \mathbf{x}$ and $\bar{\mathbf{R}} = \mathbf{Q} \cdot \bar{\mathbf{R}}$, where $\mathbf{Q} \in SO(3)$, the strain measures transform as

$$\bar{\mathbf{U}}^* = \bar{\mathbf{U}} \quad \text{and} \quad \bar{\mathbf{v}}^* = \mathbf{Q} \cdot \bar{\mathbf{v}} \cdot \mathbf{Q}^t \tag{11}$$

and we conclude that $\bar{\mathbf{v}}$ represents an objective strain measure.

Remark. Note carefully that the micropolar stretch tensors $\bar{\mathbf{U}}$ and $\bar{\mathbf{v}}$ coincide with the *classical* stretch $\mathbf{U}$ and $\mathbf{v}$ in the case of identical micropolar $\bar{\mathbf{R}}$ and continuum rotation $\mathbf{R}$, i.e. $\bar{\mathbf{R}}^t \cdot \mathbf{R} = \mathbf{I}$.

2.3. *Micropolar curvature*

The variation of the *independent* triades, i.e. the variation of the orientation of the material particles within a body is conceived as an additional measure of deformation. Therefore, we define the four possible third-order curvature tensors as

$$\overset{3}{\mathbf{K}} = \bar{\mathbf{R}}^t \cdot \nabla_X \bar{\mathbf{R}} = \operatorname{spn}(\mathbf{K}_K)\mathbf{G}^K, \quad \overset{3}{\boldsymbol{\kappa}} = \nabla_x \bar{\mathbf{R}} \cdot \bar{\mathbf{R}}^t = \operatorname{spn}(\boldsymbol{\kappa}_k)\mathbf{g}^k,$$
$$\overset{3}{\boldsymbol{\Gamma}} = \nabla_X \bar{\mathbf{R}} \cdot \bar{\mathbf{R}}^t = \operatorname{spn}(\boldsymbol{\Gamma}_K)\mathbf{G}^K, \quad \overset{3}{\boldsymbol{\gamma}} = \bar{\mathbf{R}}^t \cdot \nabla_x \bar{\mathbf{R}} = \operatorname{spn}(\gamma_k)\mathbf{g}^k, \tag{12}$$

Obviously, these curvature measures are related through $\bar{\mathbf{R}}$ and $\mathbf{F}$ by *pull-back* and *push-forward* operations, which merely changes the base vectors. It is useful to express the curvature tensors with respect to the convective base vectors

$$\overset{3}{\mathbf{K}} = \kappa_{IJK}\bar{\mathbf{G}}^I \otimes \bar{\mathbf{G}}^J \otimes \mathbf{G}^K, \quad \overset{3}{\boldsymbol{\kappa}} = \kappa_{ijk}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k$$

$$\overset{3}{\mathbf{\Gamma}} = \kappa_{ijK}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{G}^K, \quad \overset{3}{\gamma} = \kappa_{IJk}\bar{\mathbf{G}}^I \otimes \bar{\mathbf{G}}^J \otimes \mathbf{g}^k. \tag{13}$$

Since spn $(\mathbf{K}_K)$, spn $(\boldsymbol{\kappa}_k)$, spn $(\mathbf{\Gamma}_K)$ and spn (γ_k) are skew-symmetric tensors, i.e. $\kappa_{IJK} = -\kappa_{JIK}$, the third-order tensors may be reduced to second-order curvature measures

$$\mathbf{K} = \mathrm{axl}(\overset{3}{\mathbf{K}}) = \mathbf{K}_K\mathbf{G}^K = \kappa_{IJ}\bar{\mathbf{G}}^I \otimes \mathbf{G}^J, \quad \mathbf{K} = \mathrm{axl}(\overset{3}{\boldsymbol{\kappa}}) = \boldsymbol{\kappa}_k\mathbf{g}^k = \kappa_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$$

$$\mathbf{\Gamma} = \mathrm{axl}(\overset{3}{\mathbf{\Gamma}}) = \mathbf{\Gamma}_K\mathbf{G}^K = \kappa_{iJ}\mathbf{g}^i \otimes \mathbf{G}^J, \quad \gamma = \mathrm{axl}(\overset{3}{\gamma}) = \gamma_k\mathbf{g}^k = \kappa_{Ij}\bar{\mathbf{G}}^I \otimes \mathbf{g}^j. \tag{14}$$

Clearly, these curvature measures are connected as

$$\mathbf{K} = \bar{\mathbf{R}}^t \cdot \boldsymbol{\kappa} \cdot \mathbf{F} = \bar{\mathbf{R}}^t \cdot \mathbf{\Gamma} = \gamma \cdot \mathbf{F}, \quad \boldsymbol{\kappa} = \bar{\mathbf{R}} \cdot \mathbf{K} \cdot \mathbf{F}^{-1} = \mathbf{\Gamma} \cdot \mathbf{F}^{-1} = \bar{\mathbf{R}} \cdot \gamma$$

$$\mathbf{\Gamma} = \bar{\mathbf{R}} \cdot \gamma \cdot \mathbf{F} = \boldsymbol{\kappa} \cdot \mathbf{F} = \bar{\mathbf{R}} \cdot \mathbf{K}, \quad \gamma = \bar{\mathbf{R}}^t \cdot \mathbf{\Gamma} \cdot \mathbf{F}^{-1} = \mathbf{K} \cdot \mathbf{F}^{-1} = \bar{\mathbf{R}}^t \cdot \boldsymbol{\kappa}. \tag{15}$$

These relations are symbolically represented in the curvature tetrahedron in Fig. 2. The spatial curvature tensor $\boldsymbol{\kappa}$ transforms in the same manner as the spatial metric tensor. Nevertheless, the curvature does not define an invertible mapping, e.g. of line elements, in the sense of the deformation gradient or the rotation tensor.

Under a superposed rigid body motion with $\mathbf{x}^* = \mathbf{c} + \mathbf{Q} \cdot \mathbf{x}$ and $\bar{\mathbf{R}}^* = \mathbf{Q} \cdot \bar{\mathbf{R}}$, where $\mathbf{Q} \in SO(3)$, the third-order curvature measures transform as

$$\overset{3}{\mathbf{K}}{}^* = \overset{3}{\mathbf{K}} \quad \overset{3}{\boldsymbol{\kappa}}{}^* = [\mathbf{Q} \cdot \overset{3}{\boldsymbol{\kappa}} \cdot \mathbf{Q}^t] \cdot \mathbf{Q}^t \quad \overset{3}{\mathbf{\Gamma}}{}^* = \mathbf{Q} \cdot \overset{3}{\mathbf{\Gamma}} \cdot \mathbf{Q}^t \quad \overset{3}{\gamma}{}^* = [\overset{3}{\gamma}] \cdot \mathbf{Q}^t. \tag{16}$$

Therefore, the second-order representations transform as

$$\mathbf{K}^* = \mathbf{K} \quad \boldsymbol{\kappa}^* = \mathbf{Q} \cdot \boldsymbol{\kappa} \cdot \mathbf{Q}^t \quad \mathbf{\Gamma}^* = \mathbf{Q} \cdot \mathbf{\Gamma} \quad \gamma^* = \gamma \cdot \mathbf{Q}^t \tag{17}$$

and we conclude that $\boldsymbol{\kappa}$ represents an objective curvature measure. In analogy to the definition of the left micropolar stretch tensor $\bar{\mathbf{v}} = \mathbf{F} \cdot \bar{\mathbf{R}}^t$ and to formulate constitutive relations in the spatial setting we introduce the objective left curvature tensor $\mathbf{k} = \mathbf{\Gamma} \cdot \bar{\mathbf{R}}^t$ with $\mathbf{k}^* = \mathbf{Q} \cdot \mathbf{k} \cdot \mathbf{Q}^t$ in contrast to the right curvature tensor $\mathbf{K} = \bar{\mathbf{R}}^t \cdot \mathbf{k} \cdot \bar{\mathbf{R}}$.

Remark. The relations between the curvature measures $\mathbf{K}$, $\boldsymbol{\kappa}$, $\mathbf{\Gamma}$ and γ and the rotation (pseudo) vector $\boldsymbol{\theta}$ are established in Appendix A.

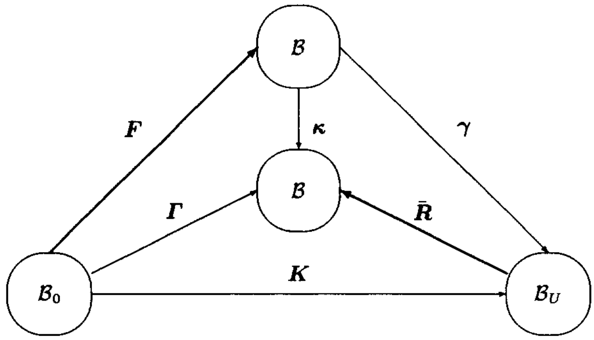

Fig. 2. Curvature tetrahedron.

3. MICROPOLAR STRESS AND COUPLE STRESS

In the case of micropolar continua the Cauchy, or true couple stress tensor $\mathbf{m}$, and couple stress vector $\mathbf{t}_m$ are defined by analogy to the Cauchy or true stress tensor $\boldsymbol{\sigma}$ and stress vector $\mathbf{t}$, respectively, via the Cauchy theorem by the linear mappings in the spatial configuration $\mathscr{B}$

$$\mathbf{t} = \boldsymbol{\sigma}^t \cdot \mathbf{n} \rightsquigarrow \boldsymbol{\sigma} \quad \text{and} \quad \mathbf{t}_m = \mathbf{m}^t \cdot \mathbf{n} \rightsquigarrow \mathbf{m}. \tag{18}$$

Here $\mathbf{n}$ denotes the normal to an element of surface area $\mathrm{d}a$ at $\mathbf{x} \in \partial\mathscr{B}$ in the spatial configuration. Due to the Cauchy postulate the Cauchy stress and couple stress vector depend exclusively on the surface normal. The Cauchy theorem follows from equilibrium of stresses and couple stresses at an infinitesimal tetrahedron element. The Cauchy stress and couple stress tensors are weighted with the determinant of the Jacobi transformation to render the Kirchhoff stress and couple stress tensors in $\mathscr{B}$ as

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} = J\sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \quad \text{and} \quad \boldsymbol{\mu} = J\mathbf{m} = Jm^{ij}\mathbf{g}_i \otimes \mathbf{g}_j. \tag{19}$$

The Nanson formula $\mathbf{n}\,\mathrm{d}a = J\mathbf{F}^{-t} \cdot \mathbf{N}\,\mathrm{d}A$ together with the configuration independency requirement of the resulting infinitesimal surface traction and torque, i.e. $\mathbf{t}\,\mathrm{d}a = \mathbf{T}\,\mathrm{d}A$ and $\mathbf{t}_m\,\mathrm{d}a = \mathbf{T}_m\,\mathrm{d}A$, allows one to refer the true stress and couple stress tensors to an element of surface area in the reference configuration $\mathrm{d}A$ at $\mathbf{X} \in \partial\mathscr{B}_0$ to yield the definition of the first Piola–Kirchhoff stress and couple stress tensors in $(\mathscr{B}_0, \mathscr{B})$

$$\boldsymbol{\Sigma}_1 = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} = J\sigma^{ij}\mathbf{G}_I \otimes \mathbf{g}_j \quad \text{and} \quad \mathbf{M}_1 = J\mathbf{F}^{-1} \cdot \mathbf{m} = Jm^{ij}\mathbf{G}_I \otimes \mathbf{g}_j. \tag{20}$$

Equivalently, two-field stress and couple stress measures in $(\mathscr{B}_U, \mathscr{B})$ follow as

$$\bar{\boldsymbol{\tau}} = \bar{\mathbf{R}}^t \cdot \boldsymbol{\tau} = J\sigma^{ij}\bar{\mathbf{G}}_I \otimes \mathbf{g}_j \quad \text{and} \quad \bar{\boldsymbol{\mu}} = \bar{\mathbf{R}}^t \cdot \boldsymbol{\mu} = Jm^{ij}\bar{\mathbf{G}}_I \otimes \mathbf{g}_j. \tag{21}$$

Finally, two-field Biot stress and couple stress measures in $(\mathscr{B}_0, \mathscr{B}_U)$ with associated rotated stress and couple stress vectors $\bar{\mathbf{T}} = \bar{\mathbf{R}}^t \cdot \mathbf{T}$ and $\bar{\mathbf{T}}_m = \bar{\mathbf{R}}^t \cdot \mathbf{T}_m$ are obtained as

$$\bar{\boldsymbol{\Sigma}}_1 = \boldsymbol{\Sigma}_1 \cdot \bar{\mathbf{R}} = J\sigma^{ij}\mathbf{G}_I \otimes \bar{\mathbf{G}}_J \quad \text{and} \quad \bar{\mathbf{M}}_1 = \mathbf{M}_1 \cdot \bar{\mathbf{R}} = Jm^{ij}\mathbf{G}_I \otimes \bar{\mathbf{G}}_J \tag{22}$$

and are given by a constitutive assumption below. The different stress and couple stress measures and their conjugate strain and curvature measures are shown in Fig. 3.

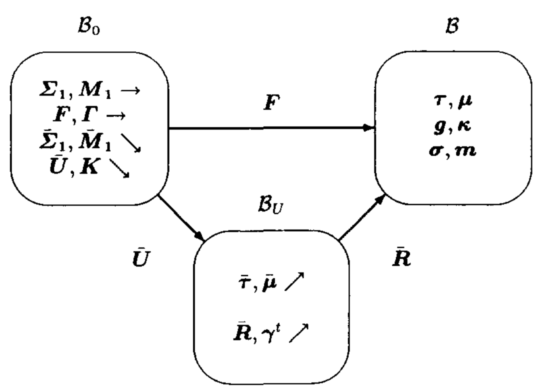

Fig. 3. Stress and couple stress measures and conjugated variables.

4. MICROPOLAR KINEMATICS OF FINITE ELASTOPLASTICITY

4.1. *Elastoplastic multiplicative decompositions*

In analogy to the *multiplicative* decomposition of the deformation gradient **F** into an elastic and a plastic contribution proposed originally by Lee (1969) within the classical Boltzmann continuum, an equivalent phenomenological decomposition is postulated for the micropolar rotation tensor $\bar{\mathbf{R}}$ within the present formulation of a micropolar continuum, i.e.

$$\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p \quad \text{and} \quad \bar{\mathbf{R}} = \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p. \tag{23}$$

Within single crystal metals the *multiplicative* decomposition of the deformation gradient is motivated micromechanically by the dislocation flow along crystalline slip systems ($\mathbf{F}_p$) followed by a distortion of the lattice ($\mathbf{F}_e$). In the context of micropolar elastoplasticity the viewpoint is adopted that the dislocation flow influences the orientation of the *independent* triade as well, thus giving rise to the notion of the plastic micropolar rotation $\bar{\mathbf{R}}_p$, which is complemented by the elastic micropolar rotation $\bar{\mathbf{R}}_e$. Phenomenologically, an unloaded stress and couple stress-free configuration is therefore locally defined by $\mathbf{F}_e^{-1}$ together with $\bar{\mathbf{R}}_e^t$. In the following, as a consequence of these considerations, the objective stress and couple stress response will depend solely on the elastic parts of the (spatial) left stretch and curvature measures. For a comparison, see the analogous discussion concerning this topic within the Boltzmann continuum by Simo and Miehe (1992).

The elastoplastic decompositions introduce a set of configurations associated with the elastic and plastic parts of the deformation gradient and the micropolar rotation. These are shown in Fig. 4. Here $\mathscr{B}_0$, $\mathscr{B}_p$ and $\mathscr{B}$ denote the reference, the plastic intermediate and the spatial configurations, respectively. Micropolar rotation with respect to the reference or the spatial configurations $\mathscr{B}_0$ or $\mathscr{B}$ introduces the (micro-)rotated configurations $\mathscr{B}_R$ and $\mathscr{B}_U$. Finally, the *multiplicative* decomposition of the micropolar rotation defines the elastically (micro-)rotated configurations $\mathscr{B}_{R_e}$ and $\mathscr{B}_{U_e}$ and the plastically (micro-)rotated configurations $\mathscr{B}_{R_p}$ and $\mathscr{B}_{U_p}$, respectively.

It has to be emphasized that no independent decomposition into an elastic and a plastic part is necessary for the micropolar curvature since the *multiplicative* decomposition of the micropolar rotation induces an *additive* structure for the curvature, e.g.

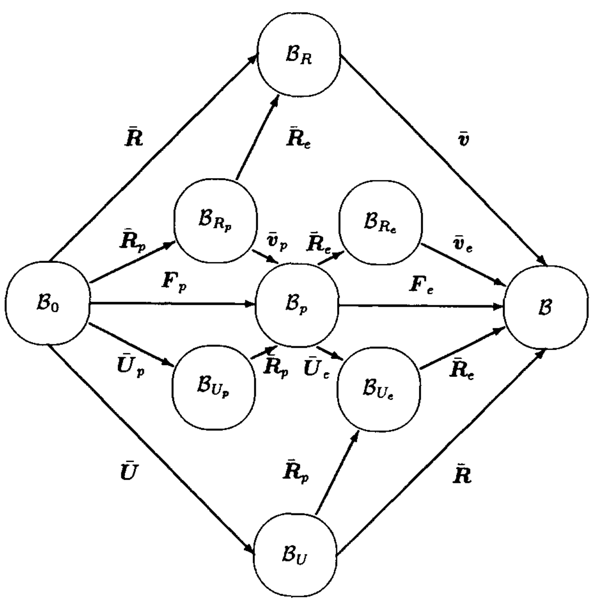

Fig. 4. Configurations associated with the multiplicative decompositions.

$$\overset{3}{\mathbf{K}} = \bar{\mathbf{R}}^t \cdot \nabla_X \bar{\mathbf{R}} = \bar{\mathbf{R}}^t \cdot \nabla_X \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p + \bar{\mathbf{R}}_p^t \cdot \nabla_X \bar{\mathbf{R}}_p. \tag{24}$$

The different strain and curvature measures associated with these configurations and their connections by micropolar *push-forward* and *pull-back* are defined in the following sections. The definitions of micropolar *push-forward* and *pull-back* are given in Appendix B.

4.2. *Micropolar strain*

A possible set of (contravariant) strain measures associated with the *multiplicative* decompositions is shown in Fig. 5. These strain measures are constructed by combining either the total, the elastic or the plastic parts of the deformation gradient $\mathbf{F}$ and the micropolar rotation $\bar{\mathbf{R}}$. Clearly, the strain measures in the different configurations are connected by micropolar *push-forward* and *pull-back*. As a paradigm consider the relation between the spatial metric $\mathbf{g}$, the elastic micropolar right stretch tensor $\bar{\mathbf{U}}_e$ and the micropolar right stretch tensor $\bar{\mathbf{U}}$

$$\begin{aligned} \mathbf{g}^{-1} &= \mathbf{\Phi}_*(\bar{\mathbf{U}}^{-1}) = \mathbf{F} \cdot \bar{\mathbf{U}}^{-1} \cdot \bar{\mathbf{R}}^t \\ &= \mathbf{\Phi}_*^e(\bar{\mathbf{U}}_e^{-1}) = \mathbf{F}_e \cdot \bar{\mathbf{U}}_e^{-1} \cdot \bar{\mathbf{R}}_e^t. \end{aligned} \tag{25}$$

However, the most important strain measures for the subsequent derivations of constitutive equations are the spatial micropolar left stretch tensor $\bar{\mathbf{v}}$ as the micropolar analogue to the Finger tensor within the *classical* continuum theory

$$\begin{aligned} \bar{\mathbf{v}} &= \mathbf{\Phi}_*(\mathbf{G}^{-1}) = \mathbf{F} \cdot \mathbf{G}^{-1} \cdot \bar{\mathbf{R}}^t \\ &= \mathbf{\Phi}_*^e(\bar{\mathbf{v}}_p) = \mathbf{F}_e \cdot \bar{\mathbf{v}}_p \cdot \bar{\mathbf{R}}_e^t \end{aligned} \tag{26}$$

and the spatial elastic micropolar left stretch tensor $\bar{\mathbf{v}}_e$ with

$$\begin{aligned} \bar{\mathbf{v}}_e &= \mathbf{\Phi}_*(\bar{\mathbf{U}}_p^{-1}) = \mathbf{F} \cdot \bar{\mathbf{U}}_p^{-1} \cdot \bar{\mathbf{R}}^t \\ &= \mathbf{\Phi}_*^e(\mathbf{G}_p^{-1}) = \mathbf{F}_e \cdot \mathbf{G}_p^{-1} \cdot \bar{\mathbf{R}}_e^t. \end{aligned} \tag{27}$$

Remark. Observe the resulting *multliplicative* structure of the micropolar strain measures, which is also typical for *multiplicative* elasoplasticity within the Boltzmann continuum.

4.3. *Micropolar curvature*

In accordance with the proposed *multiplicative* decomposition of $\bar{\mathbf{R}} = \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p$ the (covariant) curvature measures decompose *additively* into an elastic and a plastic part, consider the micropolar right or the spatial curvature tensors, for example

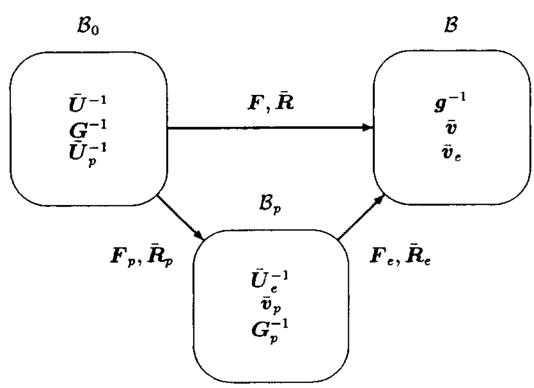

Fig. 5. Micropolar strain measures.

$$\overset{3}{\mathbf{K}} = \overset{3}{\mathbf{K}}_e + \overset{3}{\mathbf{K}}_p = \bar{\mathbf{R}}_p^t \cdot \bar{\mathbf{R}}_e^t \cdot \nabla_X \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p + \bar{\mathbf{R}}_p^t \cdot \nabla_X \bar{\mathbf{R}}_p$$

$$\overset{3}{\boldsymbol{\kappa}} = \overset{3}{\boldsymbol{\kappa}}_e + \overset{3}{\boldsymbol{\kappa}}_p = \nabla_x \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_e^t + \bar{\mathbf{R}}_e \cdot \nabla_x \bar{\mathbf{R}}_p \cdot \bar{\mathbf{R}}_p^t \cdot \bar{\mathbf{R}}_e^t, \tag{28}$$

respectively. The second-order representations of these curvature measures are connected by micropolar *push-forward* and *pull-back*

$$\mathbf{K} = \operatorname{axl}(\overset{3}{\mathbf{K}}) = \bar{\mathbf{R}}^t \cdot \boldsymbol{\kappa} \cdot \mathbf{F} = \boldsymbol{\Phi}^*(\boldsymbol{\kappa})$$

$$\mathbf{K}_e = \operatorname{axl}(\overset{3}{\mathbf{K}}_e) = \bar{\mathbf{R}}^t \cdot \boldsymbol{\kappa}_e \cdot \mathbf{F} = \boldsymbol{\Phi}^*(\boldsymbol{\kappa}_e)$$

$$\mathbf{K}_p = \operatorname{axl}(\overset{3}{\mathbf{K}}_p) = \bar{\mathbf{R}}^t \cdot \boldsymbol{\kappa}_p \cdot \mathbf{F} = \boldsymbol{\Phi}^*(\boldsymbol{\kappa}_p). \tag{29}$$

By analogy to the definition of the left micropolar stretch tensor $\bar{\mathbf{v}} = \bar{\mathbf{R}} \cdot \bar{\mathbf{U}} \cdot \bar{\mathbf{R}}^t$ the micropolar left curvature measures, as shown in Fig. 6, are introduced as

$$\mathbf{k} = \bar{\mathbf{R}} \cdot \mathbf{K} \cdot \bar{\mathbf{R}}^t = \bar{\mathbf{R}}_e \cdot \tilde{\mathbf{k}} \cdot \bar{\mathbf{R}}_e^t$$

$$\mathbf{k}_e = \bar{\mathbf{R}} \cdot \mathbf{K}_e \cdot \bar{\mathbf{R}}^t = \bar{\mathbf{R}}_e \cdot \tilde{\mathbf{k}}_e \cdot \bar{\mathbf{R}}_e^t$$

$$\mathbf{k}_p = \bar{\mathbf{R}} \cdot \mathbf{K}_p \cdot \mathbf{R}^t = \bar{\mathbf{R}}_e \cdot \tilde{\mathbf{k}}_p \cdot \mathbf{R}_e^t. \tag{30}$$

The left and the elastic left curvature measures $\mathbf{k}$ and $\mathbf{k}_e$ are essential for the subsequent derivations of the constitutive laws.

Remark. Observe the *additive* structure of the curvature measures $\mathbf{K} = \mathbf{K}_e + \mathbf{K}_p$, $\mathbf{k} = \mathbf{k}_e + \mathbf{k}_p$ and $\boldsymbol{\kappa} = \boldsymbol{\kappa}_e + \boldsymbol{\kappa}_p$ introduced by the *multiplicative* decomposition of $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$ and $\bar{\mathbf{R}} = \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p$.

5. MICROPOLAR KINEMATICAL RATES

5.1. *Micropolar strain rates*

Due to the *multiplicative* decomposition of $\mathbf{F} = \mathbf{F}_e \cdot \mathbf{F}_p$ and $\bar{\mathbf{R}} = \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p$ the spatial velocity gradient $\mathbf{l}$ and the micropolar spatial spin $\boldsymbol{\Omega}$ and its axial vector $\boldsymbol{\omega}$ decompose *additively* in $\mathscr{B}$

$$\mathbf{l} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \mathbf{l}_e + \mathbf{l}_p = \dot{\mathbf{F}}_e \cdot \mathbf{F}_e^{-1} + \mathbf{F}_e \cdot \dot{\mathbf{F}}_p \cdot \mathbf{F}_p^{-1} \cdot \mathbf{F}_e^{-1} \tag{31}$$

$$\boldsymbol{\Omega} = \dot{\bar{\mathbf{R}}} \cdot \bar{\mathbf{R}}^t = \boldsymbol{\Omega}_e + \boldsymbol{\Omega}_p = \dot{\bar{\mathbf{R}}}_e \cdot \bar{\mathbf{R}}_e^t + \bar{\mathbf{R}}_e \cdot \dot{\bar{\mathbf{R}}}_p \cdot \bar{\mathbf{R}}_p^t \cdot \bar{\mathbf{R}}_e^t \in so(3) \tag{32}$$

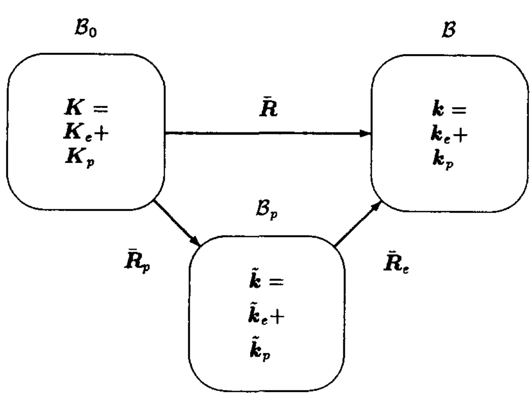

Fig. 6. Micropolar curvature measures.

$$\boldsymbol{\omega} = \operatorname{axl}(\boldsymbol{\Omega}) = \boldsymbol{\omega}_e + \boldsymbol{\omega}_p \in \mathbb{R}^3. \tag{33}$$

The material time derivative of the micropolar right stretch tensor $\bar{\mathbf{U}}$ is easily derived as

$$\dot{\bar{\mathbf{U}}} = \bar{\mathbf{R}}^t \cdot [\mathbf{l} - \boldsymbol{\Omega}] \cdot \mathbf{F}. \tag{34}$$

Applying the micropolar *push-forward* to this result renders the micropolar Lie time derivatives of the spatial metric

$$\mathscr{L}_c(\mathbf{g}) = \mathbf{l} - \boldsymbol{\Omega}. \tag{35}$$

The definition of micropolar Lie time derivatives is given in Appendix B. The material time derivative of the spatial micropolar left stretch tensor $\bar{\mathbf{v}}$ defined in $\mathscr{B}$ is obtained as

$$\dot{\bar{\mathbf{v}}} = \mathscr{L}_c(\mathbf{g}) \cdot \bar{\mathbf{v}} + [\bar{\mathbf{v}} \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \bar{\mathbf{v}}]. \tag{36}$$

Finally, we derive the material time derivative of the spatial micropolar elastic left stretch tensor $\bar{\mathbf{v}}_e$ defined in $\mathscr{B}$ as

$$\dot{\bar{\mathbf{v}}}_e = \mathscr{L}_c(\mathbf{g}) \cdot \bar{\mathbf{v}}_e + \mathscr{L}_c(\bar{\mathbf{v}}_e) + [\bar{\mathbf{v}}_e \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \bar{\mathbf{v}}_e]. \tag{37}$$

Remark. Note carefully that the micropolar Lie derivative of the spatial metric coincides formally with the structure of the micropolar strain measure used by Günther (1958) in the geometrically linear micropolar continuum $\boldsymbol{\varepsilon}^{\text{lin}} = \nabla_x \mathbf{u} - \operatorname{spn}(\boldsymbol{\omega})$, where $\mathbf{u}$ and $\boldsymbol{\omega}$ are the infinitesimal displacement and rotation vectors.

Remark. Observe that neither the spatial angular velocity vector $\boldsymbol{\omega}$ nor the material angular velocity vector $\hat{\boldsymbol{\omega}} = \bar{\mathbf{R}}^t \cdot \boldsymbol{\omega}$ can be integrated, since in general they are not an exact differential form and therefore cannot be regarded as the rate of change of any vector. Exceptions to this rule are rotations about a fixed axis.

Remark. The angular velocities $\boldsymbol{\omega}$ and $\hat{\boldsymbol{\omega}}$ obey the vectorial characteristics of the transformation and the parallelogram rule. In contradiction, the rotation $\boldsymbol{\theta}$ is a (pseudo) vector since the parallelogram rule for two subsequent rotations $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ is not valid. The resulting rotation (pseudo) vector $\boldsymbol{\theta}_3$ of two subsequent rotations $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ follows as

$$\bar{\boldsymbol{\theta}}_3 = \frac{\bar{\boldsymbol{\theta}}_2 + \bar{\boldsymbol{\theta}}_1 + \operatorname{spn}(\bar{\boldsymbol{\theta}}_2) \cdot \bar{\boldsymbol{\theta}}_1}{1 - \bar{\boldsymbol{\theta}}_2 \cdot \bar{\boldsymbol{\theta}}_1}, \quad \bar{\boldsymbol{\theta}} = \frac{\boldsymbol{\theta}}{\theta} \tan\frac{\theta}{2} \tag{38}$$

which is a result of the underlying multiplicative update for the associated rotation tensors $\bar{\mathbf{R}}_3 = \bar{\mathbf{R}}_2 \cdot \bar{\mathbf{R}}_1$.

5.2. *Micropolar curvature rates*

Due to the *multiplicative* decomposition of $\bar{\mathbf{R}} = \bar{\mathbf{R}}_e \cdot \bar{\mathbf{R}}_p$ the spatial gradient of the micropolar spatial angular velocity vector $\nabla_x \boldsymbol{\omega}$ decomposes *additively* in $\mathscr{B}$

$$\nabla_x \boldsymbol{\omega} = \nabla_x(\operatorname{axl}(\boldsymbol{\Omega})) = \nabla_x \boldsymbol{\omega}_e + \nabla_x \boldsymbol{\omega}_p. \tag{39}$$

Since the micropolar spatial spin $\boldsymbol{\Omega} \in \operatorname{so}(3)$ is skew-symmetric, the material time derivatives of the third-order material right curvature tensor $\overset{3}{\mathbf{K}}$ and its second-order representation $\mathbf{K}$ are derived as

$$(\overset{3}{\mathbf{K}})^{\cdot} = \bar{\mathbf{R}}^t \cdot \nabla_X \boldsymbol{\Omega} \cdot \bar{\mathbf{R}} \rightarrow \dot{\mathbf{K}} = \bar{\mathbf{R}}^t \cdot \nabla_x \boldsymbol{\omega} \cdot \mathbf{F}. \tag{40}$$

Equivalently, the micropolar Lie time derivative of the spatial curvature follows as

$$\mathscr{L}_c(\boldsymbol{\kappa}) = \nabla_x \boldsymbol{\omega} \tag{41}$$

with *additive* decomposition into elastic and plastic parts

$$\mathscr{L}_c(\boldsymbol{\kappa}_e) = \nabla_x \boldsymbol{\omega}_e - \boldsymbol{\Omega}_p \cdot \boldsymbol{\kappa}_e \quad \text{and} \quad \mathscr{L}_c(\boldsymbol{\kappa}_p) = \nabla_x \boldsymbol{\omega}_p + \boldsymbol{\Omega}_p \cdot \boldsymbol{\kappa}_e. \tag{42}$$

The material time derivative of the spatial left curvature tensor $\mathbf{k}$ defined in $\mathscr{B}$ is obtained as

$$\dot{\mathbf{k}} = \mathscr{L}_c(\boldsymbol{\kappa}) \cdot \bar{\mathbf{v}} + [\mathbf{k} \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \mathbf{k}]. \tag{43}$$

Finally, the material time derivative of the spatial elastic left curvature tensor $\mathbf{k}_e$ defined in $\mathscr{B}$ is derived as

$$\dot{\mathbf{k}}_e = \mathscr{L}_c(\boldsymbol{\kappa}_e) \cdot \bar{\mathbf{v}} + [\mathbf{k}_e \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \mathbf{k}_e]. \tag{44}$$

Remark. Observe the *additive* structure of all rates of the curvature measures, e.g. $\dot{\mathbf{k}} = \dot{\mathbf{k}}_e + \dot{\mathbf{k}}_p$, etc.

Remark. The micropolar Lie derivative of the spatial curvature coincides formally with the structure of the micropolar curvature considered by Günther (1958) in the geometrically linear micropolar continuum $\boldsymbol{\kappa}^{\text{lin}} = \nabla_x \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the infinitesimal rotation vector.

6. MICROPOLAR STRESS AND COUPLE STRESS RATES

The rates of the stress and couple stress measures are important for the linearization of the weak form of the balance equations in the material $\mathscr{B}_0$ and the spatial configuration $\mathscr{B}$ and are therefore connected to the fourth-order Lagrange and Euler material operators. The material rates of the first Piola–Kirchhoff stress and couple stress tensors $\dot{\boldsymbol{\Sigma}}_1$ and $\dot{\mathbf{M}}_1$ governing the incremental balance equations in the reference configuration are computed as

$$\dot{\boldsymbol{\Sigma}}_1 = \mathscr{L}_c(\boldsymbol{\Sigma}_1) + \boldsymbol{\Sigma}_1 \cdot \boldsymbol{\Omega}^t \quad \text{and} \quad \dot{\mathbf{M}}_1 = \mathscr{L}_c(\mathbf{M}_1) + \mathbf{M}_1 \cdot \boldsymbol{\Omega}^t. \tag{45}$$

In the case of micropolar hyperelasticity with stored energy function $W = \rho_0 \Psi$ the Lie derivatives of the first Piola–Kirchhoff stress and couple stress tensors follow as

$$\begin{aligned} \mathscr{L}_c(\boldsymbol{\Sigma}_1^t) &= {}^{\sigma\sigma}\mathscr{L} : [\dot{\mathbf{F}} - \boldsymbol{\Omega} \cdot \mathbf{F}] + {}^{\sigma m}\mathscr{L} : \nabla_X \boldsymbol{\omega}, \\ \mathscr{L}_c(\mathbf{M}_1^t) &= {}^{m\sigma}\mathscr{L} : [\dot{\mathbf{F}} - \boldsymbol{\Omega} \cdot \mathbf{F}] + {}^{mm}\mathscr{L} : \nabla_X \boldsymbol{\omega} \end{aligned} \tag{46}$$

with the Lagrange material operators

$$\begin{aligned} {}^{\sigma\sigma}\mathscr{L} &= \bar{\mathbf{R}} \cdot \partial^2_{\bar{\mathbf{U}}\bar{\mathbf{U}}} W \cdot \bar{\mathbf{R}}^t, \quad {}^{\sigma m}\mathscr{L} = \bar{\mathbf{R}} \cdot \partial^2_{\bar{\mathbf{U}}\mathbf{K}} W \cdot \bar{\mathbf{R}}^t \\ {}^{m\sigma}\mathscr{L} &= \bar{\mathbf{R}} \cdot \partial^2_{\mathbf{K}\bar{\mathbf{U}}} W \cdot \bar{\mathbf{R}}^t, \quad {}^{mm}\mathscr{L} = \bar{\mathbf{R}} \cdot \partial^2_{\mathbf{K}\mathbf{K}} W \cdot \bar{\mathbf{R}}^t. \end{aligned} \tag{47}$$

On the other hand the nominal stress and couple stress rates defined as

$$\mathring{\boldsymbol{\sigma}} = \mathscr{L}_c(\boldsymbol{\sigma}) + \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}^t \quad \text{and} \quad \mathring{\mathbf{m}} = \mathscr{L}_c(\mathbf{m}) + \mathbf{m} \cdot \boldsymbol{\Omega}^t \tag{48}$$

govern the incremental balance equations in the spatial setting and are identical to the *push-forward* of

$$\dot{\Sigma}_{\text{l}}^{t} = \frac{\mathrm{D}}{\mathrm{D}t}(J\boldsymbol{\sigma}^{t}\cdot\mathbf{F}^{-t}) \quad \text{and} \quad \dot{\mathbf{M}}_{\text{l}}^{t} = \frac{\mathrm{D}}{\mathrm{D}t}(J\mathbf{m}^{t}\cdot\mathbf{F}^{-t}). \tag{49}$$

The nominal rates are connected to the material rates via

$$\mathring{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{l}\cdot\boldsymbol{\sigma} + [\mathbf{l}:\mathbf{g}]\boldsymbol{\sigma} \quad \text{and} \quad \mathring{\mathbf{m}} = \dot{\mathbf{m}} - \mathbf{l}\cdot\mathbf{m} + [\mathbf{l}:\mathbf{g}]\mathbf{m}. \tag{50}$$

Equivalently, the Truesdell rates are defined as

$$\begin{aligned} \mathscr{L}_{\text{c}}(\boldsymbol{\sigma}) &= \dot{\boldsymbol{\sigma}} - \mathbf{l}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{\Omega}^{t} + [\mathbf{l}:\mathbf{g}]\boldsymbol{\sigma}, \\ \mathscr{L}_{\text{c}}(\mathbf{m}) &= \dot{\mathbf{m}} - \mathbf{l}\cdot\mathbf{m} - \mathbf{m}\cdot\boldsymbol{\Omega}^{t} + [\mathbf{l}:\mathbf{g}]\mathbf{m} \end{aligned} \tag{51}$$

and are related to the Oldroyd rates with $\dot{\boldsymbol{\tau}} = J\dot{\boldsymbol{\sigma}} + [\mathbf{l}:\mathbf{g}]\boldsymbol{\tau}$ and $\dot{\boldsymbol{\mu}} = J\dot{\mathbf{m}} + [\mathbf{l}:\mathbf{g}]\boldsymbol{\mu}$ through

$$\begin{aligned} \mathscr{L}_{\text{c}}(\boldsymbol{\tau}) &= J\mathscr{L}_{\text{c}}(\boldsymbol{\sigma}) = \dot{\boldsymbol{\tau}} - \mathbf{l}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\boldsymbol{\Omega}^{t}, \\ \mathscr{L}_{\text{c}}(\boldsymbol{\mu}) &= J\mathscr{L}_{\text{c}}(\mathbf{m}) = \dot{\boldsymbol{\mu}} - \mathbf{l}\cdot\boldsymbol{\mu} - \boldsymbol{\mu}\cdot\boldsymbol{\Omega}^{t}. \end{aligned} \tag{52}$$

In the case of isotropic micropolar hyperelasticity the Lie derivatives of the Cauchy stresses and couple stresses rendering the objective Truesdell stress rate in the case of the Boltzmann continuum are determined as

$$\begin{aligned} \mathscr{L}_{\text{c}}(\boldsymbol{\sigma}^{t}) &= {}^{\sigma\sigma}\mathscr{E}:[\mathbf{l}-\boldsymbol{\Omega}] + {}^{\sigma m}\mathscr{E}:\nabla_{x}\boldsymbol{\omega} \\ \mathscr{L}_{\text{c}}(\mathbf{m}^{t}) &= {}^{m\sigma}\mathscr{E}:[\mathbf{l}-\boldsymbol{\Omega}] + {}^{mm}\mathscr{E}:\nabla_{x}\boldsymbol{\omega}. \end{aligned} \tag{53}$$

Here the Euler material operators follow either as the *push-forward* of the Lagrange material operators or in analogy to the representation in Miehe (1993) within the Boltzmann continuum as

$$\begin{aligned} {}^{\sigma\sigma}\mathscr{E} &= J^{-1}\bar{\mathbf{v}}\cdot\partial^{2}_{\bar{\mathbf{v}}\bar{\mathbf{v}}}W\cdot\bar{\mathbf{v}}^{t}, \quad & {}^{\sigma m}\mathscr{E} &= J^{-1}\bar{\mathbf{v}}\cdot\partial^{2}_{\bar{\mathbf{v}}\mathbf{k}}W\cdot\bar{\mathbf{v}}^{t} \\ {}^{m\sigma}\mathscr{E} &= J^{-1}\bar{\mathbf{v}}\cdot\partial^{2}_{\mathbf{k}\bar{\mathbf{v}}}W\cdot\bar{\mathbf{v}}^{t}, \quad & {}^{mm}\mathscr{E} &= J^{-1}\bar{\mathbf{v}}\cdot\partial^{2}_{\mathbf{k}\mathbf{k}}W\cdot\bar{\mathbf{v}}^{t}. \end{aligned} \tag{54}$$

7. MICROPOLAR CONSTITUTIVE EQUATIONS

7.1. *Micropolar hyperelasticity*

In the case of purely mechanical micropolar hyperelasticity the sum of the stress and couple stress power $\boldsymbol{\tau}^{t}:\mathscr{L}_{\text{c}}(\mathbf{g})$ and $\boldsymbol{\mu}^{t}:\mathscr{L}_{\text{c}}(\boldsymbol{\kappa})$ equals the change in free Helmholtz energy Ψ, i.e.

$$\boldsymbol{\tau}^{t}:[\mathbf{l}-\boldsymbol{\Omega}] + \boldsymbol{\mu}^{t}:\nabla_{x}\boldsymbol{\omega} - \rho_{0}\dot{\Psi} = 0. \tag{55}$$

The Helmholtz free energy is assumed to be an isotropic scalar valued tensor function of the deformation and curvature measures, i.e.

$$\Psi = \Psi(\bar{\mathbf{U}}, \mathbf{K}, \mathbf{G}^{-1}) = \Psi(\bar{\mathbf{v}}, \mathbf{k}, \mathbf{g}). \tag{56}$$

Then the change in Helmholtz free energy follows as

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\bar{\mathbf{U}}}:\dot{\bar{\mathbf{U}}} + \frac{\partial\Psi}{\partial\mathbf{K}}:\dot{\mathbf{K}} = \frac{\partial\Psi}{\partial\bar{\mathbf{v}}}:\mathring{\bar{\mathbf{v}}} + \frac{\partial\Psi}{\partial\mathbf{k}}:\mathring{\mathbf{k}} \tag{57}$$

with the micropolar strain and curvature rates

$$\mathring{\bar{\mathbf{U}}} = \bar{\mathbf{R}}^t \cdot \mathscr{L}_c(\mathbf{g}) \cdot \mathbf{F}, \quad \mathring{\bar{\mathbf{v}}} = \mathscr{L}_c(\mathbf{g}) \cdot \bar{\mathbf{v}} + [\bar{\mathbf{v}} \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \bar{\mathbf{v}}] \tag{58}$$

$$\mathring{\mathbf{K}} = \bar{\mathbf{R}}^t \cdot \mathscr{L}_c(\boldsymbol{\kappa}) \cdot \mathbf{F}, \quad \mathring{\mathbf{k}} = \mathscr{L}_c(\boldsymbol{\kappa}) \cdot \bar{\mathbf{v}} + [\mathbf{k} \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \mathbf{k}]. \tag{59}$$

Following standard arguments the elastic constitutive equations for the spatial stress and couple stress are derived

$$\boldsymbol{\tau}^t = \rho_0 \bar{\mathbf{R}} \cdot \frac{\partial \Psi}{\partial \bar{\mathbf{U}}} \cdot \mathbf{F}^t = \rho_0 \frac{\partial \Psi}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t \quad \text{and} \quad \boldsymbol{\mu}^t = \rho_0 \bar{\mathbf{R}} \cdot \frac{\partial \Psi}{\partial \mathbf{K}} \cdot \mathbf{F}^t = \rho_0 \frac{\partial \Psi}{\partial \mathbf{k}} \cdot \bar{\mathbf{v}}^t, \tag{60}$$

where the following holds due to isotropy

$$\left[\bar{\mathbf{v}}^t \cdot \frac{\partial \Psi}{\partial \bar{\mathbf{v}}} - \frac{\partial \Psi}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t\right] : \boldsymbol{\Omega}^t = 0 \quad \text{and} \quad \left[\mathbf{k}^t \cdot \frac{\partial \Psi}{\partial \mathbf{k}} - \frac{\partial \Psi}{\partial \mathbf{k}} \cdot \mathbf{k}^t\right] : \boldsymbol{\Omega}^t = 0. \tag{61}$$

Remark. Note carefully the analogy to the determination of the Kirchhoff stress in terms of the spatial Finger tensor $\mathbf{b}$ in the case of a hyperelastic Boltzmann continuum $\boldsymbol{\tau} = \rho_0 \partial_{\mathbf{b}} \Psi \cdot \mathbf{b}$ advocated in Truesdell and Noll (1965).

7.2. *Micropolar stored energy function*

Without loss of generality it is assumed that the micropolar stored energy function may be decoupled into separate strain energy and curvature energy contributions

$$W = \rho_0 \Psi = \rho_0 {}^s\Psi + \rho_0 {}^c\Psi = {}^sW + {}^cW, \tag{62}$$

where ${}^sW$ and ${}^cW$ are isotropic scalar functions of a nonsymmetric tensor valued argument. Due to the representation theorems by Wang (1969) these functions are expressed by sets of irreducible basic invariants

$$\begin{aligned} {}^sI_n^{\text{sym}} &= [\bar{\mathbf{v}}^{\text{sym}}]^n : \mathbf{I}, \quad {}^sI_2^{\text{skw}} = [\bar{\mathbf{v}}^{\text{skw}}]^2 : \mathbf{I}, \quad {}^sI_n^{\text{mix}} = [[\bar{\mathbf{v}}^{\text{sym}}]^n \cdot [\bar{\mathbf{v}}^{\text{skw}}]^2] : \mathbf{I} \\ {}^cI_n^{\text{sym}} &= [\mathbf{k}^{\text{sym}}]^n : \mathbf{I}, \quad {}^cI_2^{\text{skw}} = [\mathbf{k}^{\text{skw}}]^2 : \mathbf{I}, \quad {}^cI_n^{\text{mix}} = [[\mathbf{k}^{\text{sym}}]^n \cdot [\mathbf{k}^{\text{skw}}]^2] : \mathbf{I} \end{aligned} \tag{63}$$

with the symmetric and skew-symmetric parts of the nonsymmetric micropolar left stretch and curvature tensors $\bar{\mathbf{v}}$ and $\mathbf{k}$. Then the stored energy functions are represented by

$$\begin{aligned} {}^sW &= {}^sW(\bar{\mathbf{v}}^{\text{sym}}, \bar{\mathbf{v}}^{\text{skw}}) = {}^sW({}^sI_1^{\text{sym}}, {}^sI_2^{\text{sym}}, {}^sI_3^{\text{sym}}, {}^sI_2^{\text{skw}}, {}^sI_1^{\text{mix}}, {}^sI_2^{\text{mix}}) \\ {}^cW &= {}^cW(\mathbf{k}^{\text{sym}}, \mathbf{k}^{\text{skw}}) = {}^cW({}^cI_1^{\text{sym}}, {}^cI_2^{\text{sym}}, {}^cI_3^{\text{sym}}, {}^cI_2^{\text{skw}}, {}^cI_1^{\text{mix}}, {}^cI_2^{\text{mix}}). \end{aligned} \tag{64}$$

In the following the explicit dependence of the strain energy function on the mixed invariants is omitted for the sake of simplicity. With these preliminaries at hand, the Kirchhoff stress and couple stress associated with a given stored energy function are evaluated as

$$\begin{aligned} \boldsymbol{\tau}^t &= \frac{\partial {}^sW}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t = \frac{\partial {}^sW}{\partial {}^sI_n^{\text{sym}}} \frac{\partial {}^sI_n^{\text{sym}}}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t + \frac{\partial {}^sW}{\partial {}^sI_2^{\text{skw}}} \frac{\partial {}^sI_2^{\text{skw}}}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t \\ \boldsymbol{\mu}^t &= \frac{\partial {}^cW}{\partial \mathbf{k}} \cdot \bar{\mathbf{v}}^t = \frac{\partial {}^cW}{\partial {}^cI_n^{\text{sym}}} \frac{\partial {}^cI_n^{\text{sym}}}{\partial \mathbf{k}} \cdot \bar{\mathbf{v}}^t + \frac{\partial {}^cW}{\partial {}^cI_2^{\text{skw}}} \frac{\partial {}^cI_2^{\text{skw}}}{\partial \mathbf{k}} \cdot \bar{\mathbf{v}}^t. \end{aligned} \tag{65}$$

Remark. The derivatives of the basic invariants ${}^sI_n^{\text{sym}}$ and ${}^sI_2^{\text{skw}}$ with respect to their arguments are computed as

$$\frac{\partial\, {}^s I_n^{\rm sym}}{\partial \bar{\mathbf{v}}} = n[\bar{\mathbf{v}}^{\rm sym}]^{n-1}, \quad \frac{\partial\, {}^s I_2^{\rm skw}}{\partial \bar{\mathbf{v}}} = -2\bar{\mathbf{v}}^{\rm skw}. \tag{66}$$

For the invariants under consideration we have

$$\frac{\partial\, {}^s I_1^{\rm sym}}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t = \bar{\mathbf{v}}^t \tag{67}$$

$$\frac{\partial\, {}^s I_2^{\rm sym}}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t = [\bar{\mathbf{v}}^2 + \mathbf{b}]^t \tag{68}$$

$$\frac{\partial\, {}^s I_3^{\rm sym}}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t = \tfrac{3}{4}[\mathbf{b}\cdot\bar{\mathbf{v}}^t + \mathbf{b}\cdot\bar{\mathbf{v}} + \bar{\mathbf{v}}\cdot\mathbf{b} + \bar{\mathbf{v}}^3]^t \tag{69}$$

$$\frac{\partial\, {}^s I_2^{\rm skw}}{\partial \bar{\mathbf{v}}} \cdot \bar{\mathbf{v}}^t = [\bar{\mathbf{v}}^2 - \mathbf{b}]^t, \tag{70}$$

where we introduced the symmetric Finger tensor $\mathbf{b} = \bar{\mathbf{v}}\cdot\bar{\mathbf{v}}^t = \mathbf{F}\cdot\mathbf{F}^t$ which tends to $\bar{\mathbf{v}}^2$ if the micropolar rotation $\bar{\mathbf{R}}$ tends to the continuum rotation $\mathbf{R}$.

Example. A simple *planar* micropolar hyperelastic model is represented by the extension of a compressible volumetric deviatoric decoupled neo-Hooke material to the micropolar case. To derive constitutive models with decoupled volumetric deviatoric behaviour for the stress part we follow the approach advocated in Simo *et al.* (1985) within the nonpolar continuum. To this end, the left stretch tensor $\bar{\mathbf{v}}$ is decomposed *multiplicatively* into an isochoric and a volumetric contribution

$$\bar{\mathbf{v}} = J^{1/3}\hat{\mathbf{v}} \rightsquigarrow \hat{\mathbf{v}} = J^{-1/3}\bar{\mathbf{v}} \quad \text{with} \quad J = \det\mathbf{F} = \det\bar{\mathbf{v}}. \tag{71}$$

For the derivative of the isochoric part $\hat{\mathbf{v}}$ with respect to $\bar{\mathbf{v}}$ we have

$$\frac{\partial\hat{\mathbf{v}}}{\partial\bar{\mathbf{v}}} = J^{-1/3}[\mathscr{I} - \tfrac{1}{3}\bar{\mathbf{v}}\otimes\bar{\mathbf{v}}^{-t}]. \tag{72}$$

If we construct the strain energy function in terms of the isochoric strains we obtain

$$\frac{\partial\, {}^s\hat{I}_1}{\partial\bar{\mathbf{v}}}\cdot\bar{\mathbf{v}}^t = \hat{\mathbf{v}}^t - \frac{{}^s\hat{I}_1}{3}\mathbf{I} = \operatorname{dev}\hat{\mathbf{v}}^t = \operatorname{dev}\hat{\mathbf{v}}^{\rm sym} - \hat{\mathbf{v}}^{\rm skw} \quad \text{with} \quad {}^s\hat{I}_1 = J^{-1/3}\,{}^s I_1^{\rm sym}. \tag{73}$$

In accordance with the representation theorems for scalar isotropic functions with nonsymmetric tensor argument, the strain energy function is then chosen as

$${}^sW = {}^sW^{\rm dev} + {}^sW^{\rm vol} \quad \text{with} \quad {}^sW^{\rm dev} = 2\mu[{}^s\hat{I}_1 - 3] \quad \text{and} \quad {}^sW^{\rm vol} = U(J). \tag{74}$$

Here μ may be regarded as the usual shear modulus and the convex function $U(J)$ with $U(1) = 0$ contains at least one material parameter, e.g. the bulk modulus. For the curvature part of the constitutive equations within the two-dimensional case the curvature measures may be represented by vectors and the simplest curvature energy function is given with the additional material parameter l

$${}^cW = \mu l^2\, {}^cI_2 = \mu l^2 \mathbf{k}\cdot\mathbf{k} \quad \text{with} \quad \mathbf{k}\in\mathbb{R}^2. \tag{75}$$

The experimental determination of the additional constant l is still an open question. Nevertheless, the solution of inverse problems for parameter identification of mechanical systems is a rapidly growing area of research. Then the Kirchhoff stress and couple stress tensors follow as

$$\begin{aligned} \boldsymbol{\tau} &= 2\mu[\operatorname{dev} \hat{\mathbf{v}}^{\mathrm{sym}} + \hat{\mathbf{v}}^{\mathrm{skw}}] + JU'\mathbf{I} \in GL(3) \\ \boldsymbol{\mu} &= 2\mu l^2 \mathbf{k} \cdot \bar{\mathbf{v}}^t \qquad \in \mathbb{R}^2. \end{aligned} \tag{76}$$

Finally, the Euler material operators are computed according to eqn (54) as

$$\begin{aligned} J^{\sigma\sigma}\mathscr{E} &= \tfrac{2}{3}\mu^s \hat{I}_1[\mathscr{I} - \tfrac{1}{3}\mathbf{I} \otimes \mathbf{I}] - \tfrac{4}{3}\mu[\operatorname{dev} \hat{\mathbf{v}}^t \otimes \mathbf{I}]^{\mathrm{sym}} + J^2 U''\mathbf{I} \otimes \mathbf{I} + JU'[\mathbf{I} \otimes \mathbf{I} - \mathscr{I}] \\ J^{mm}\mathscr{E} &= 2\mu l^2 \mathbf{b}. \end{aligned} \tag{77}$$

7.3. *Micropolar flow rule within finite elastoplasticity*

The reduced form of the Clausius–Duhem inequality for the purely mechanical micropolar continuum theory consists of the stress and couple stress power $\boldsymbol{\tau}^t : \mathscr{L}_c(\mathbf{g})$ and $\boldsymbol{\mu}^t : \mathscr{L}_c(\boldsymbol{\kappa})$ minus the change in free Helmholtz energy $\dot{\psi}$. Therefore, the extension of the Clausius–Duhem inequality within the Boltzmann continuum to the micropolar case is expressed as

$$\boldsymbol{\tau}^t : [\mathbf{l} - \boldsymbol{\Omega}] + \boldsymbol{\mu}^t : \nabla_x \boldsymbol{\omega} - \rho_0 \dot{\Psi} \geqslant 0. \tag{78}$$

To describe perfect plasticity the Helmholtz free energy is assumed to depend exclusively on the elastic parts of the spatial deformation and curvature measures, i.e.

$$\Psi = \Psi(\bar{\mathbf{v}}_e, \mathbf{k}_e, \mathbf{g}). \tag{79}$$

Then the change in Helmholtz free energy follows as

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \bar{\mathbf{v}}_e} : \dot{\bar{\mathbf{v}}}_e + \frac{\partial \Psi}{\partial \mathbf{k}_e} : \dot{\mathbf{k}}_e \tag{80}$$

with the material rates of the elastic spatial strain and curvature measures

$$\dot{\bar{\mathbf{v}}}_e = \mathscr{L}_c(\mathbf{g}) \cdot \bar{\mathbf{v}}_e + \mathscr{L}_c(\bar{\mathbf{v}}_e) + [\bar{\mathbf{v}}_e \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \bar{\mathbf{v}}_e] \tag{81}$$

$$\dot{\mathbf{k}}_e = \mathscr{L}_c(\boldsymbol{\kappa}) \cdot \bar{\mathbf{v}} - \mathscr{L}_c(\boldsymbol{\kappa}_p) \cdot \bar{\mathbf{v}} + [\mathbf{k}_e \cdot \boldsymbol{\Omega}^t - \boldsymbol{\Omega}^t \cdot \mathbf{k}_e]. \tag{82}$$

Following standard arguments the elastic constitutive equations are derived as

$$\boldsymbol{\tau}^t = \rho_0 \frac{\partial \Psi}{\partial \bar{\mathbf{v}}_e} \cdot \bar{\mathbf{v}}_e^t \quad \text{and} \quad \boldsymbol{\mu}^t = \rho_0 \frac{\partial \Psi}{\partial \mathbf{k}_e} \cdot \mathbf{v}^t, \tag{83}$$

where the following holds due to isotropy

$$\left[\bar{\mathbf{v}}_e^t \cdot \frac{\partial \Psi}{\partial \bar{\mathbf{v}}_e} - \frac{\partial \Psi}{\partial \bar{\mathbf{v}}_e} \cdot \bar{\mathbf{v}}_e^t\right] : \boldsymbol{\Omega}^t = 0 \quad \text{and} \quad \left[\mathbf{k}_e^t \cdot \frac{\partial \Psi}{\partial \mathbf{k}_e} - \frac{\partial \Psi}{\partial \mathbf{k}_e} \cdot \mathbf{k}_e^t\right] : \boldsymbol{\Omega}^t = 0. \tag{84}$$

Then the dissipation inequality $\mathbb{R}^9 \times \mathbb{R}^9 \to \mathbb{R}_+$ remains as

$$-\boldsymbol{\tau}^t : [\mathscr{L}_c(\bar{\mathbf{v}}_e) \cdot \bar{\mathbf{v}}_e^{-1}] + \boldsymbol{\mu}^t : \mathscr{L}_c(\boldsymbol{\kappa}_p) \geqslant 0. \tag{85}$$

The elastic domain in Kirchhoff stress–couple stress space is defined by

$$\mathbb{E} := \{(\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathbb{R}^9 \times \mathbb{R}^9 | \Phi(\boldsymbol{\tau}, \boldsymbol{\mu}) \leqslant 0\}. \tag{86}$$

Here the Kirchhoff stresses and couple stresses are restricted by a single yield condition which is represented in accordance with proposals, for example, in Besdo (1974) or de Borst (1991) by an isotropic convex scalar valued tensor function

$$\Phi(\boldsymbol{\tau}, \boldsymbol{\mu}) : \mathbb{R}^9 \times \mathbb{R}^9 \rightarrow \mathbb{R}. \tag{87}$$

Following standard procedure the principle of maximum dissipation for the micropolar case

$$-[\boldsymbol{\tau}^t - \tilde{\boldsymbol{\tau}}^t] : [\mathscr{L}_c(\bar{\mathbf{v}}_e) \cdot \bar{\mathbf{v}}_e^{-1}] + [\boldsymbol{\mu}^t - \tilde{\boldsymbol{\mu}}^t] : \mathscr{L}_c(\boldsymbol{\kappa}_p) \geqslant 0 \quad \forall\ (\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\mu}}) \in \mathbb{E} \tag{88}$$

renders the associated flow rules for the plastic strain and curvature rates

$$\mathscr{L}_c(\bar{\mathbf{v}}_e) \cdot \bar{\mathbf{v}}_e^{-1} = -\lambda \frac{\partial \Phi}{\partial \boldsymbol{\tau}^t} \quad \text{and} \quad \mathscr{L}_c(\boldsymbol{\kappa}_p) = \lambda \frac{\partial \Phi}{\partial \boldsymbol{\mu}^t}. \tag{89}$$

Here the Lagrange parameter may be interpreted as the plastic multiplier. Equivalently, the flow rules are represented in the reference configuration as

$$\dot{\bar{\mathbf{U}}}_p^{-1} \cdot \bar{\mathbf{U}}_p = -\lambda \mathbf{F}^{-1} \cdot \frac{\partial \Phi}{\partial \boldsymbol{\tau}^t} \cdot \mathbf{F} \quad \text{and} \quad \dot{\mathbf{K}}_p = \lambda \bar{\mathbf{R}}^t \cdot \frac{\partial \Phi}{\partial \boldsymbol{\mu}^t} \cdot \mathbf{F}. \tag{90}$$

Remark. The extension to incorporate isotropic hardening is straightforward. To this end, the argument list of the Helmholtz free energy is extended by a scalar strain-like variable describing pointwise defects such as dislocation pile ups, etc., and the conjugate thermodynamic force is considered within the yield condition.

Remark. Note the identical structure for the strain part of the flow rule and for the flow rule developed by Simo and Miehe (1992) within the *classical* continuum if the *nonsymmetric* micropolar stretch $\bar{\mathbf{v}}_e$ is substituted by the *symmetric* elastic Finger tensor $\mathbf{b}_e$. Observe further the *additive* structure of the curvature part in contrast to the *multiplicative* structure of the strain part of the flow rule.

Remark. Embedding the approach of Lippmann (1969), Besdo (1974), Mühlhaus and Vardoulakis (1987) or de Borst (1991) into the current framework the yield condition within *planar* micropolar von Mises plasticity may be defined as

$$\Phi = \sqrt{|\text{dev}\ \boldsymbol{\tau}^{\text{sym}}|^2 + \frac{1}{l^2}|\boldsymbol{\mu}|^2} - \sqrt{\tfrac{2}{3}}\,Y = \varphi - \sqrt{\tfrac{2}{3}}\,Y \leqslant 0 \quad \text{with} \quad \boldsymbol{\mu} \in \mathbb{R}^2, \tag{91}$$

where dev $\boldsymbol{\tau}^{\text{sym}}$ denotes the symmetric part of the Kirchhoff stress deviator, l is a material constant and Y denotes the yield limit. Then the associated flow rules follow as

$$\mathscr{L}_c(\bar{\mathbf{v}}_e) \cdot \bar{\mathbf{v}}_e^{-1} = -\lambda \frac{\text{dev}\ \boldsymbol{\tau}^{\text{sym}}}{\varphi} \quad \text{and} \quad \mathscr{L}_c(\boldsymbol{\kappa}_p) = \lambda \frac{\boldsymbol{\mu}}{l^2 \varphi}. \tag{92}$$

8. BALANCE EQUATIONS

The *continuity equation* follows as in the *classical* continuum by requiring the conservation of the total mass of a body with $\mathrm{d}m_0 = \rho_0\,\mathrm{d}V$ and $\mathrm{d}m = \rho\,\mathrm{d}v$ rendering immediately the result $\rho_0 = J\rho$ since $\mathrm{d}v = J\,\mathrm{d}V$. The material time derivative of the total mass yields the material and with $\dot{J} = J[\mathbf{1} : \mathbf{g}]$ the spatial representation of the continuity equation as local form of mass conservation

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathscr{B}_0} \mathrm{d}m_0 = \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathscr{B}} \mathrm{d}m = 0 \rightsquigarrow \dot{\rho}_0 = \dot{\rho}+\rho[\mathbf{1}:\mathbf{g}] = 0. \tag{93}$$

In the following the *balance of linear momentum* is formulated. Thereby, the first Cauchy equation of motion is the consequence of postulating that during deformation the change of linear momentum is balanced with the external forces. Therefore, the postulate of balance of linear momentum is expressed as

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathscr{B}_0} \dot{\mathbf{x}}\,\mathrm{d}m_0 = \int_{\partial\mathscr{B}_0} \mathbf{T}\,\mathrm{d}A+\int_{\partial\mathscr{B}_0} \mathbf{B}\,\mathrm{d}m_0 \quad \text{or} \quad \frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathscr{B}} \dot{\mathbf{x}}\,\mathrm{d}m = \int_{\partial\mathscr{B}} \mathbf{t}\,\mathrm{d}a+\int_{\mathscr{B}} \mathbf{B}\,\mathrm{d}m. \tag{94}$$

Here $\mathbf{T}$ and $\mathbf{t}$ denote the stress vectors referred to $\mathrm{d}A$ and $\mathrm{d}a$, and $\mathbf{B}$ is the vector of body force per unit mass. Applying the continuity equation, the Gauss theorem together with the Cauchy theorem $\mathbf{T} = \boldsymbol{\Sigma}'_1 \cdot \mathbf{N}$ and $\mathbf{t} = \boldsymbol{\sigma}' \cdot \mathbf{n}$ results in the local form of the first Cauchy equation of motion

$$\rho_0\ddot{\mathbf{x}} = \mathrm{Div}\,\boldsymbol{\Sigma}'_1+\rho_0\mathbf{B} \quad \text{or} \quad \rho\ddot{\mathbf{x}} = \mathrm{div}\,\boldsymbol{\sigma}'+\rho\mathbf{B}. \tag{95}$$

Neglecting inertia terms and body forces, the incremental translational equilibrium is expressed as

$$\mathbf{o} = \mathrm{Div}\,\dot{\boldsymbol{\Sigma}}'_1 \quad \text{or} \quad \mathbf{o} = \mathrm{div}\,\mathring{\boldsymbol{\sigma}}'. \tag{96}$$

Next, the *balance of angular momentum* for the micropolar continuum is derived. To this end, it is postulated that the angular momentum of a body defined as the integral of the cross product $\overset{3}{\mathbf{e}} : [\mathbf{r}\otimes\dot{\mathbf{x}}]$ with $\mathbf{r} = \mathbf{x}-\mathbf{x}_0$ and the spin $\mathbf{s} = \mathfrak{J}\cdot\boldsymbol{\omega}$ varies exclusively due to an external torque. Therefore, the postulate of balance of angular momentum results in

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathscr{B}_0} [\overset{3}{\mathbf{e}}: [\mathbf{r}\otimes\dot{\mathbf{x}}]+\mathbf{s}]\,\mathrm{d}m_0 = \int_{\partial\mathscr{B}_0} [\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\mathbf{T}]+\mathbf{T}_m]\,\mathrm{d}A+\int_{\mathscr{B}_0} [\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\mathbf{B}]+\mathbf{B}_m]\,\mathrm{d}m_0$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{\mathscr{B}} [\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\dot{\mathbf{x}}]+\mathbf{s}]\,\mathrm{d}m = \int_{\partial\mathscr{B}} [\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\mathbf{t}]+\mathbf{t}_m]\,\mathrm{d}a+\int_{\mathscr{B}} [\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\mathbf{B}]+\mathbf{B}_m]\,\mathrm{d}m. \tag{97}$$

Here $\mathbf{T}_m = \mathbf{M}'_1\cdot\mathbf{N}$ and $\mathbf{t}_m = \mathbf{m}'\cdot\mathbf{n}$ denote the couple stress vectors referred to $\mathrm{d}A$ and $\mathrm{d}a$, $\mathbf{B}_m$ is the vector of body couple per unit mass and $\mathfrak{J}$ is the second-order inertia tensor of microrotation per unit mass. Using the continuity equation, the relation $\overset{3}{\mathbf{e}}:[\dot{\mathbf{r}}\otimes\dot{\mathbf{x}}] = \mathbf{o}$, the Gauss theorem, the Cauchy theorem $\mathbf{T} = \boldsymbol{\Sigma}'_1\cdot\mathbf{N}$, $\mathbf{T}_m = \mathbf{M}'_1\cdot\mathbf{N}$ or equivalently $\mathbf{t} = \boldsymbol{\sigma}'\cdot\mathbf{m}$, $\mathbf{t}_m = \mathbf{m}'\cdot\mathbf{n}$, together with the relation $\mathrm{Div}(\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\boldsymbol{\Sigma}'_1]) = \overset{3}{\mathbf{e}}:[\boldsymbol{\tau}+\mathbf{r}\otimes\mathrm{Div}\,\boldsymbol{\Sigma}'_1]$ or equivalently $\mathrm{div}(\overset{3}{\mathbf{e}}:[\mathbf{r}\otimes\boldsymbol{\sigma}']) = \overset{3}{\mathbf{e}}:[\boldsymbol{\sigma}+\mathbf{r}\otimes\mathrm{div}\,\boldsymbol{\sigma}']$ and the first Cauchy equation of motion, the postulate results in local form as

$$\rho_0\dot{\mathbf{s}} = \overset{3}{\mathbf{e}}:\boldsymbol{\tau}+\mathrm{Div}\,\mathbf{M}'_1+\rho_0\mathbf{B}_m \quad \text{or} \quad \rho\dot{\mathbf{s}} = \overset{3}{\mathbf{e}}:\boldsymbol{\sigma}+\mathrm{div}\,\mathbf{m}'+\rho\mathbf{B}_m. \tag{98}$$

Neglecting rotational inertia and body couples the incremental rotational equilibrium is expressed as

$$\mathbf{o} = \overset{3}{\mathbf{e}} : \boldsymbol{\tau} + \mathrm{Div}\, \dot{\mathbf{M}}_1^t \quad \text{or} \quad \mathbf{o} = \overset{3}{\mathbf{e}} : [\dot{\boldsymbol{\sigma}} + \mathrm{div}\, \mathbf{v}\boldsymbol{\sigma}] + \mathrm{div}\, \dot{\mathbf{m}}^t. \tag{99}$$

Spin, couple stresses and body couples are not present in the classical continuum with only three translational degrees of freedom for every particle. They are typical ingredients of micropolar continua, e.g. the Cosserat continuum. For the nonpolar case the second Cauchy equation of motion, the so-called Boltzmann axiom, follows as the symmetry condition for the Cauchy stress tensor $\boldsymbol{\sigma}^{\mathrm{skw}} = \mathbf{o}$.

9. DIRICHLET VARIATIONAL PRINCIPLE

It is the aim of this section to demonstrate that the stationarity of the total potential energy $\Pi(\mathbf{x}, \boldsymbol{\theta})$ in the framework of hyperelastic micropolar continua

$$\delta\Pi = \int_{\mathscr{B}_0} \delta W \,\mathrm{d}V + \delta\Pi^{\mathrm{ext}} = \int_{\mathscr{B}_0} \left[\delta\bar{\mathbf{U}} : \frac{\partial W}{\partial \bar{\mathbf{U}}} + \delta\mathbf{K} : \frac{\partial W}{\partial \mathbf{K}} \right] \mathrm{d}V + \delta\Pi^{\mathrm{ext}} = 0 \tag{100}$$

is equivalent to the weak form of the balance of linear and angular momentum together with Neumann boundary conditions $\mathbf{T}^p = \boldsymbol{\Sigma}_1^t \cdot \mathbf{N}$ on $\partial\mathscr{B}_0^T$ and $\mathbf{T}_m^p = \mathbf{M}_1^t \cdot \mathbf{N}$ on $\partial\mathscr{B}_0^{T_m}$. It is assumed that the *conservative* loading results in the following representation for the first variation of the external potential energy

$$\delta\Pi^{\mathrm{ext}} = -\int_{\partial\mathscr{B}_0^T} \delta\mathbf{u} \cdot \mathbf{T}^p \,\mathrm{d}A - \int_{\partial\mathscr{B}^{T_m}} \tilde{\boldsymbol{\omega}} \cdot \mathbf{T}_m^p \,\mathrm{d}A. \tag{101}$$

Here variations are performed with respect to the position vector $\mathbf{x}$ and the (pseudo) rotation vector $\boldsymbol{\theta}$ rendering $\delta\bar{\mathbf{U}} = \bar{\mathbf{R}}^t \cdot [\tilde{\boldsymbol{\Omega}}^t \cdot \mathbf{F} + \delta\mathbf{F}]$, $\delta\mathbf{K} = \bar{\mathbf{R}}^t \cdot \nabla_X \tilde{\boldsymbol{\omega}}$ and $\delta\bar{\mathbf{R}} = \tilde{\boldsymbol{\Omega}} \cdot \bar{\mathbf{R}}$. Neglecting translational inertia and body forces the weak form of the balance of linear momentum follows for all test functions $\delta\mathbf{u}$ satisfying $\delta\mathbf{u} = \mathbf{o}$ on $\partial\mathscr{B}_0^u$ and using the boundary conditions $\mathbf{T}^p = \mathbf{M}_1^t \cdot \mathbf{N}$ on $\partial\mathscr{B}_0^T$ as

$$\int_{\mathscr{B}_0} \nabla_X^t \delta\mathbf{u} : \boldsymbol{\Sigma}_1 \,\mathrm{d}V = \int_{\partial\mathscr{B}_0^T} \delta\mathbf{u} \cdot \mathbf{T}^p \,\mathrm{d}A \quad \text{or} \quad \int_{\mathscr{B}} \nabla_x^t \delta\mathbf{u} : \boldsymbol{\sigma} \,\mathrm{d}v = \int_{\partial\mathscr{B}^T} \delta\mathbf{u} \cdot \mathbf{t}^p \,\mathrm{d}a. \tag{102}$$

In analogy, neglecting rotational inertia and body couples the weak form of the balance of angular momentum follows for all test functions (axial vectors) $\tilde{\boldsymbol{\omega}} = \mathrm{axl}\,(\delta\bar{\mathbf{R}} \cdot \bar{\mathbf{R}}^t)$ satisfying $\tilde{\boldsymbol{\omega}} = \mathbf{o}$ on $\partial\mathscr{B}_0^\omega$ and using the boundary conditions $\mathbf{T}_m^p = \mathbf{M}_1^t \cdot \mathbf{N}$ on $\partial\mathscr{B}_0^{T_m}$ as

$$\int_{\mathscr{B}_0} [\nabla_X^t \tilde{\boldsymbol{\omega}} : \mathbf{M}_1 - [\mathbf{F}^t \cdot \tilde{\boldsymbol{\Omega}}^t] : \boldsymbol{\Sigma}_1] \,\mathrm{d}V = \int_{\partial\mathscr{B}^{T_m}} \tilde{\boldsymbol{\omega}} \cdot \mathbf{T}_m^p \,\mathrm{d}A$$

or

$$\int_{\mathscr{B}} [\nabla_x^t \tilde{\boldsymbol{\omega}} : \mathbf{m} - \tilde{\boldsymbol{\Omega}}^t : \boldsymbol{\sigma}] \,\mathrm{d}v = \int_{\partial\mathscr{B}^{T_m}} \tilde{\boldsymbol{\omega}} \cdot \mathbf{t}_m^p \,\mathrm{d}a. \tag{103}$$

Summation of these results renders the weak form of the balance of linear and angular momentum which is equivalent to the stationarity of $\Pi(\mathbf{x}, \boldsymbol{\theta})$, e.g. in the spatial setting

$$\int_{\mathscr{B}_0} [[\nabla_x \delta\mathbf{u} - \tilde{\boldsymbol{\Omega}}]^t : \boldsymbol{\tau} + \nabla_x^t \tilde{\boldsymbol{\omega}} : \boldsymbol{\mu}] \,\mathrm{d}V = \int_{\partial\mathscr{B}^T} \delta\mathbf{u} \cdot \mathbf{t}^p \,\mathrm{d}a + \int_{\partial\mathscr{B}^{T_m}} \tilde{\boldsymbol{\omega}} \cdot \mathbf{t}_m^p \,\mathrm{d}a. \tag{104}$$

Next, the second variation of the total potential energy within the framework of micropolar continua renders the exact linearization of the generalized Dirichlet variational principle with major symmetry of the tangent operator and is given by

$$\Delta\delta\Pi = \int_{\mathscr{B}_0}\left[\delta\bar{\mathbf{U}}:\left[\frac{\partial^2 W}{\partial\bar{\mathbf{U}}\bar{\mathbf{U}}}:\Delta\bar{\mathbf{U}}+\frac{\partial^2 W}{\partial\bar{\mathbf{U}}\mathbf{K}}:\Delta\mathbf{K}\right]+\frac{\partial W}{\partial\bar{\mathbf{U}}}:\Delta\delta\bar{\mathbf{U}}\right]\mathrm{d}V$$
$$+\int_{\mathscr{B}_0}\left[\delta\mathbf{K}:\left[\frac{\partial^2 W}{\partial\mathbf{K}\mathbf{K}}:\Delta\mathbf{K}+\frac{\partial^2 W}{\partial\mathbf{K}\bar{\mathbf{U}}}:\Delta\bar{\mathbf{U}}\right]+\frac{\partial W}{\partial\mathbf{K}}:\Delta\delta\mathbf{K}\right]\mathrm{d}V. \quad (105)$$

Recall that variations are performed with respect to the position vector $\mathbf{x}$ and the (pseudo) rotation vector $\boldsymbol{\theta}$. Therefore, inserting

$$\Delta\delta\bar{\mathbf{U}} = \bar{\mathbf{R}}^t\cdot[[\tilde{\boldsymbol{\Omega}}\cdot\tilde{\boldsymbol{\Omega}}]^{\mathrm{sym}}\cdot\mathbf{F}+\tilde{\boldsymbol{\Omega}}^t\cdot\Delta\mathbf{F}+\tilde{\boldsymbol{\Omega}}^t\cdot\delta\mathbf{F}]$$
$$2\Delta\delta\mathbf{K} = \bar{\mathbf{R}}^t\cdot[\tilde{\boldsymbol{\Omega}}^t\cdot\nabla_x\tilde{\boldsymbol{\omega}}+\tilde{\boldsymbol{\Omega}}^t\cdot\nabla_x\tilde{\boldsymbol{\omega}}] \quad (106)$$

with $\Delta\delta\bar{\mathbf{R}} = [\tilde{\boldsymbol{\Omega}}\cdot\tilde{\boldsymbol{\Omega}}]^{\mathrm{sym}}\cdot\bar{\mathbf{R}}$, it is easy to prove that the Hesse matrix associated with the Dirichlet variational principle generalized to the micropolar case possesses major symmetries and is given in the spatial setting by

$$\Delta\delta\Pi = \int_{\mathscr{B}_0}[\nabla_x\delta\mathbf{u}-\tilde{\boldsymbol{\Omega}}]:[{}^{\sigma\sigma}\mathscr{E}:[\nabla_x\Delta\mathbf{u}-\tilde{\boldsymbol{\Omega}}]+{}^{\sigma m}\mathscr{E}:\nabla_x\tilde{\boldsymbol{\omega}}]J\,\mathrm{d}V$$
$$+\int_{\mathscr{B}_0}\boldsymbol{\tau}^t:[[\tilde{\boldsymbol{\Omega}}\cdot\tilde{\boldsymbol{\Omega}}]^{\mathrm{sym}}+\tilde{\boldsymbol{\Omega}}^t\cdot\nabla_x\Delta\mathbf{u}+\tilde{\boldsymbol{\Omega}}^t\cdot\nabla_x\delta\mathbf{u}]\,\mathrm{d}V$$
$$+\int_{\mathscr{B}_0}\nabla_x\tilde{\boldsymbol{\omega}}:[{}^{mm}\mathscr{E}:\nabla_x\tilde{\boldsymbol{\omega}}+{}^{m\sigma}\mathscr{E}:[\nabla_x\Delta\mathbf{u}-\tilde{\boldsymbol{\Omega}}]]J\,\mathrm{d}V$$
$$+\int_{\mathscr{B}_0}\tfrac{1}{2}\boldsymbol{\mu}^t:[\tilde{\boldsymbol{\Omega}}^t\cdot\nabla_x\tilde{\boldsymbol{\omega}}+\tilde{\boldsymbol{\Omega}}^t\cdot\nabla_x\tilde{\boldsymbol{\omega}}]\,\mathrm{d}V. \quad (107)$$

Remark. Note the decomposition of the Hesse matrix into the *material* and the *geometric* contributions, which is typical for all geometrical nonlinear theories.

Remark. Motivated by the above results the incremental variational or virtual power principle may be expressed as

$$\int_{\mathscr{B}_0}[[\nabla_x\delta\mathbf{u}-\tilde{\boldsymbol{\Omega}}]:\dot{\boldsymbol{\tau}}^t-\mathring{\tilde{\boldsymbol{\Omega}}}:\boldsymbol{\tau}^t+\nabla_x\tilde{\boldsymbol{\omega}}:\dot{\boldsymbol{\mu}}^t+\mathring{\nabla}_x\tilde{\boldsymbol{\omega}}:\boldsymbol{\mu}^t]\,\mathrm{d}V+\dot{G}^{\mathrm{ext}} = 0 \quad (108)$$

with nominal spin rates $\mathring{\tilde{\boldsymbol{\Omega}}} = [\tilde{\boldsymbol{\Omega}}\cdot\tilde{\boldsymbol{\Omega}}]^{\mathrm{skw}}+\tilde{\boldsymbol{\Omega}}\cdot\nabla_x\Delta\mathbf{u}$ and $2\mathring{\nabla}_x\tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{\Omega}}\cdot\nabla_x\tilde{\boldsymbol{\omega}}-\tilde{\boldsymbol{\Omega}}\cdot\nabla_x\tilde{\boldsymbol{\omega}}$.

Remark. The symmetry of the Hesse matrix for nonlinear conservative mechanical systems, which are equipped with a configuration space involving the rotation group SO(3), has been shown using different arguments by Simo (1992) and Bufler (1993).

Remark. The connection between the variation of the rotation tensor $\delta\bar{\mathbf{R}} = \mathrm{spn}\,(\boldsymbol{\omega})\cdot\bar{\mathbf{R}}$ and the underlying variation of the rotation (pseudo) vector $\boldsymbol{\theta}$ is motivated in Appendix C.

10. SUMMARY

The main thrust of this paper was the formulation of a geometrically exact theory of finite deformation and finite rotation micropolar elastoplasticity to elaborate a theoretical background for the numerical implementation of a generalized continuum framework involving *independent* rotational degrees of freedom which will be pursued in a forthcoming paper. The basic motivation has been provided by the success of the geometrically linear micropolar continuum description in regularizing the pathological mesh size dependence of localization computations where shear failure mechanisms play a dominant role. This concept has been extended to the geometrically nonlinear regime by introducing an enhanced configuration space consisting of the *classical* deformation map together with an *independent* rotation field. The kinematics of finite micropolar elastoplasticity are then postulated as *multiplicative* decompositions of the deformation gradient and the *independent* rotation tensor into an elastic and a plastic part, thus allowing the introduction of a set of different strain and curvature measures. The spatial strain and curvature measures are conceived as the arguments of a stored energy function governing the hyperelastic response of a micropolar continuum. By exploiting the Clausius–Duhem inequality together with the postulate of maximum dissipation associated flow rules for the plastic parts of the micropolar strain and curvature are derived. Emphasis is placed on the remarkable structures of these flow rules which mirror the underlying elastoplastic *multiplicative* decomposition. These results, together with the embedding of the theory within a variational principle, provide the basis for a numerical implementation into a nonlinear solution scheme which will constitute the natural continuation of the current research.

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APPENDIX A

To relate the curvature tensors to the rotation (pseudo) vector $\boldsymbol{\theta}$ introduce the modified rotation (pseudo) vector

$$\bar{\boldsymbol{\theta}} = \frac{\boldsymbol{\theta}}{\theta}\tan\frac{\theta}{2} \quad \text{with} \quad \bar{\theta} = \tan\frac{\theta}{2}. \tag{A1}$$

Then the Euler–Rodrigues formula is reformulated as

$$\exp(\operatorname{spn}(\boldsymbol{\theta})) = \mathbf{I} + \frac{2}{1+\bar{\theta}^2}[\operatorname{spn}(\bar{\boldsymbol{\theta}}) + \operatorname{spn}^2(\bar{\boldsymbol{\theta}})] \tag{A2}$$

and the gradient of the orthogonal tensor $\bar{\mathbf{R}} = \exp(\operatorname{spn}(\boldsymbol{\theta}))$ follows as

$$\nabla_X\bar{\mathbf{R}} = \frac{2}{1+\bar{\theta}^2}\left[\operatorname{spn}(\nabla_X\bar{\boldsymbol{\theta}}) + \nabla_X(\operatorname{spn}^2(\bar{\boldsymbol{\theta}})) - \frac{2}{1+\bar{\theta}^2}[\operatorname{spn}(\bar{\boldsymbol{\theta}}) + \operatorname{spn}^2(\bar{\boldsymbol{\theta}})] \otimes \bar{\boldsymbol{\theta}}\cdot\nabla_X\bar{\boldsymbol{\theta}}\right]. \tag{A3}$$

Straightforward calculations render the material third-order curvature tensor

$$\overset{3}{\mathbf{K}} = \frac{2}{1+\bar{\theta}^2}[\operatorname{spn}(\nabla_X\bar{\boldsymbol{\theta}}) + \operatorname{spn}(\nabla_X\bar{\boldsymbol{\theta}})\cdot\operatorname{spn}(\bar{\boldsymbol{\theta}}) - \operatorname{spn}(\bar{\boldsymbol{\theta}})\cdot\operatorname{spn}(\nabla_X\bar{\boldsymbol{\theta}})] \tag{A4}$$

together with its second-order representation

$$\mathbf{K} = \frac{2}{1+\bar{\theta}^2}[\mathbf{I} - \operatorname{spn}(\bar{\boldsymbol{\theta}})]\cdot\nabla_X\bar{\boldsymbol{\theta}} \rightsquigarrow \gamma = \frac{2}{1+\bar{\theta}^2}[\mathbf{I} - \operatorname{spn}(\bar{\boldsymbol{\theta}})]\cdot\nabla_x\bar{\boldsymbol{\theta}}. \tag{A5}$$

In analogy the spatial third-order curvature tensor is expressed as

$$\overset{3}{\boldsymbol{\kappa}} = \frac{2}{1+\bar{\theta}^2}[\operatorname{spn}(\nabla_x\bar{\boldsymbol{\theta}}) + \operatorname{spn}(\bar{\boldsymbol{\theta}})\cdot\operatorname{spn}(\nabla_x\bar{\boldsymbol{\theta}}) - \operatorname{spn}(\nabla_x\bar{\boldsymbol{\theta}})\cdot\operatorname{spn}(\bar{\boldsymbol{\theta}})] \tag{A6}$$

together with its second-order representation

$$\boldsymbol{\kappa} = \frac{2}{1+\bar{\theta}^2}[\mathbf{I} + \operatorname{spn}(\bar{\boldsymbol{\theta}})]\cdot\nabla_x\bar{\boldsymbol{\theta}} \rightsquigarrow \boldsymbol{\Gamma} = \frac{2}{1+\bar{\theta}^2}[\mathbf{I} + \operatorname{spn}(\bar{\boldsymbol{\theta}})]\cdot\nabla_X\bar{\boldsymbol{\theta}}. \tag{A7}$$

Finally the gradient of the modified rotation (pseudo) vector $\bar{\boldsymbol{\theta}}$ may be expressed in terms of the rotation (pseudo) vector $\boldsymbol{\theta}$ as

$$\nabla_X\bar{\boldsymbol{\theta}} = \frac{\bar{\theta}}{\theta}\left[\mathbf{I} + \frac{\theta - \sin\theta}{\sin\theta}\frac{\boldsymbol{\theta}\otimes\boldsymbol{\theta}}{\theta^2}\right]\cdot\nabla_X\boldsymbol{\theta}. \tag{A8}$$

Example. Define the standard basis $\mathbf{E}_I$ in $\mathbb{R}^3$ and consider an in-plane rotation about the $\mathbf{E}_3$ axis with $\boldsymbol{\theta} = \theta\mathbf{E}_3$. Then

$$\nabla_X\bar{\boldsymbol{\theta}} = \frac{1+\bar{\theta}^2}{2}\mathbf{E}_3\otimes\nabla_X\theta$$

and the second-order curvature measures $\mathbf{K}$ and $\boldsymbol{\Gamma} = \boldsymbol{\kappa}\cdot\mathbf{F}$ have the matrix representation relative to $\mathbf{E}_I$

$$\bar{\mathbf{R}} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \rightsquigarrow \mathbf{K} = \boldsymbol{\Gamma} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \theta_{,1} & \theta_{,2} & \theta_{,3}\end{bmatrix}. \tag{A9}$$

Then the curvature measures may be reduced to a vector representation with the following properties

$$\mathbf{K} = \mathbf{\Gamma} = \nabla_X \theta \rightsquigarrow \boldsymbol{\kappa} = \boldsymbol{\gamma} = \nabla_x \theta, \quad \mathbf{k} = \nabla_X \theta \cdot \bar{\mathbf{R}}^t = \nabla_x \theta \cdot \bar{\mathbf{v}}. \tag{A10}$$

APPENDIX B

Definition. Micropolar *push-forward* or *pull-back* induces a change of tensor base vectors

$$\begin{aligned}
&\mathbf{\Phi}_*(\cdot), \mathbf{\Phi}^*(\cdot) \rightsquigarrow \mathbf{G}^I;\ \bar{\mathbf{G}}^I \leftrightarrow \mathbf{g}^i;\ \mathbf{g}^i, \quad \mathbf{G}_I;\ \bar{\mathbf{G}}_I \leftrightarrow \mathbf{g}_i;\ \mathbf{g}_i\\
&\mathbf{\Phi}^e_*(\cdot), \mathbf{\Phi}^*_e(\cdot) \rightsquigarrow \mathbf{G}^I_p;\ \bar{\mathbf{G}}^I_p \leftrightarrow \mathbf{g}^i;\ \mathbf{g}^i, \quad \mathbf{G}^p_I;\ \bar{\mathbf{G}}^p_I \leftrightarrow \mathbf{g}_i;\ \mathbf{g}_i\\
&\mathbf{\Phi}^p_*(\cdot), \mathbf{\Phi}^*_p(\cdot) \rightsquigarrow \mathbf{G}^I;\ \bar{\mathbf{G}}^I \leftrightarrow \mathbf{G}^I_p;\ \bar{\mathbf{G}}^I_p, \quad \mathbf{G}_I;\ \bar{\mathbf{G}}_I \leftrightarrow \mathbf{G}^p_I;\ \bar{\mathbf{G}}^p_I
\end{aligned} \tag{B1}$$

while tensor coordinates remain constant. Micropolar *push-forward* and *pull-back* may be expressed in short form: $\mathbf{\Phi}_*(\cdot)$, $\mathbf{\Phi}^e_*(\cdot)$ and $\mathbf{\Phi}^p_*(\cdot)$ denote micropolar *push-forward* and $\mathbf{\Phi}^*(\cdot)$, $\mathbf{\Phi}^*_e(\cdot)$ and $\mathbf{\Phi}^*_p(\cdot)$ denote micropolar *pull-back* of tensorial objects with $\mathbf{F}$; $\bar{\mathbf{R}}$, $\mathbf{F}_e$; $\bar{\mathbf{R}}_e$ and $\mathbf{F}_p$; $\bar{\mathbf{R}}$, respectively.

Definition. We denote the material time derivative by $(\cdot)^\cdot = \mathrm{D}/\mathrm{D}t(\cdot)$ and the micropolar Lie time derivative of spatial objects by $\mathscr{L}_c(\cdot)$. The micropolar Lie time derivative is performed by applying the micropolar *pull-back* $\mathbf{\Phi}^*(\cdot)$ performing the material time derivative $\mathrm{D}/\mathrm{D}t(\cdot)$ and finally applying the micropolar *push-forward* $\mathbf{\Phi}_*(\cdot)$ to the spatial configuration

$$\mathscr{L}_c(\cdot) = \mathbf{\Phi}_*\left(\frac{\mathrm{D}}{\mathrm{D}t}(\mathbf{\Phi}^*(\cdot))\right). \tag{B2}$$

Obviously, the micropolar Lie time derivative denotes the time derivative of the tensor coordinates while the base vectors remain unchanged.

APPENDIX C

The variation of the rotation tensor $\delta\bar{\mathbf{R}} = \mathrm{spn}(\boldsymbol{\omega}) \cdot \bar{\mathbf{R}}$ denotes the underlying variation of the rotation (pseudo) vector $\boldsymbol{\theta}$. For a motivation reparametrize the Euler–Rodrigues formula

$$\exp(\mathrm{spn}(\boldsymbol{\theta})) = \bar{\mathbf{R}} = [\bar{q}^2 - \mathbf{q}\cdot\mathbf{q}]\mathbf{I} + 2\mathbf{q}\otimes\mathbf{q} + 2\bar{q}\,\mathrm{spn}(\mathbf{q}) \tag{C1}$$

and the associated update formula for the rotation (pseudo) vector

$$\mathbf{q}_3 = \bar{q}_1\mathbf{q}_2 + \bar{q}_2\mathbf{q}_1 + \mathrm{spn}(\mathbf{q}_2)\cdot\mathbf{q}_1, \quad \bar{q}_3 = \bar{q}_1\bar{q}_2 - \mathbf{q}_1\cdot\mathbf{q}_2 \tag{C2}$$

in terms of unit quaternions

$$\mathbf{q} + \bar{q} = \frac{\boldsymbol{\theta}}{\theta}\sin\frac{\theta}{2} + \cos\frac{\theta}{2} \in \mathbb{R}^3 \times \mathbb{R}. \tag{C3}$$

Then superpose an infinitesimal rotation with axial vector $\varepsilon\boldsymbol{\omega}$ onto a finite rotation with rotation (pseudo) vector $\boldsymbol{\theta}$ and associated unit quaternions $\mathbf{q}(\boldsymbol{\theta}) + \bar{q}(\boldsymbol{\theta})$ to obtain a one-parameter family of configurations

$$\mathbf{q}_\varepsilon = \bar{q}\mathbf{q}(\varepsilon\boldsymbol{\omega}) + \bar{q}(\varepsilon\boldsymbol{\omega})\mathbf{q} + \mathrm{spn}(\mathbf{q}(\varepsilon\boldsymbol{\omega}))\cdot\mathbf{q}, \quad \bar{q}_\varepsilon = \bar{q}\bar{q}(\varepsilon\boldsymbol{\omega}) - \mathbf{q}\cdot\mathbf{q}(\varepsilon\boldsymbol{\omega}). \tag{C4}$$

The directional derivative formula $\mathrm{d}_\varepsilon\mathbf{q}_\varepsilon|_{\varepsilon=0}$ and $\mathrm{d}_\varepsilon\bar{q}_\varepsilon|_{\varepsilon=0}$ renders the variations $\delta\mathbf{q}$ and $\delta\bar{q}$ of $\mathbf{q}$ and $\bar{q}$

$$\delta\mathbf{q} = \tfrac{1}{2}[\bar{q}\mathbf{I} - \mathrm{spn}(\mathbf{q})]\cdot\boldsymbol{\omega}, \quad \delta\bar{q} = \tfrac{1}{2}\mathbf{q}\cdot\boldsymbol{\omega} \tag{C5}$$

and finally the variation of the rotation tensor $\bar{\mathbf{R}}$ follows as

$$\delta\bar{\mathbf{R}} = \frac{\partial\bar{\mathbf{R}}}{\partial\mathbf{q}}\cdot\delta\mathbf{q} + \frac{\partial\bar{\mathbf{R}}}{\partial\bar{q}}\delta\bar{q} = \mathrm{spn}(\boldsymbol{\omega})\cdot\bar{\mathbf{R}}. \tag{C6}$$